

Torsors for Difference Algebraic Groups

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ABSTRACT

We introduce a cohomology set for groups defined by algebraic difference equations and show that it classifies torsors under the group action. This allows us to compute all torsors for large classes of groups. We also develop some tools for difference algebraic geometry and present an application to the Galois theory of differential equations depending on a discrete parameter.

Introduction

Groups defined by algebraic difference equations have been studied in connection with applications such as the Manin-Mumford conjecture (see e.g. [Hru01], [KP07]). More recently, Galois theories where the Galois groups are defined by algebraic difference equations have been developed ([DVHW14], [OW15]). In these Galois theories, the solution rings, when interpreted geometrically, are torsors under the action of the Galois group. It is therefore highly desirable to obtain a thorough understanding of the structure of torsors for groups defined by difference equations.

For algebraic groups, a starting point in the quest to understand the torsors under the action of a given algebraic group G over a field k is the canonical bijection between the (non-abelian) cohomology set $H^1(k, G)$ and the isomorphism classes of G -torsors. In this article we introduce a cohomology set $H_\sigma^1(k, G)$ for a group G defined by algebraic difference equations with coefficients in a difference field k , i.e., k is a field equipped with an endomorphism $\sigma: k \rightarrow k$. We show that indeed $H_\sigma^1(k, G)$ classifies G -torsors. This allows us to explicitly describe all G -torsors for large classes of examples.

As an application to the Galois theory of linear differential equations depending on a discrete parameter, we show that our cohomology set can also be used to classify the solutions rings (i.e., the σ -Picard-Vessiot rings) for a linear differential equation depending on a discrete parameter. Further applications to this Galois theory relating to the structure of the solution rings and the inverse problem will be presented in a future paper.

Under the heading “constrained cohomology” E. Kolchin introduced in [Kol86] (see also [Kov77]) a cohomology set for groups defined by algebraic differential equations that serves a similar purpose as our cohomology set for groups defined by algebraic difference equations. The approach taken by Kolchin is in spirit close to the Galois cohomology for algebraic groups and the term “constrained” refers to the fact that cocycles are certain maps from the automorphism group of the constrained closure (nowadays more commonly referred to as a “differential closure”) of the base differential field into the group.

For a given difference field k , it seems that so far no object that could play the role of a “difference closure of k ” or of its automorphism group has been found. We completely avoid this issue by employing a functorial approach, somewhat in the spirit of how faithfully flat descent generalizes and replaces Galois descent. In particular, for us, cocycles for a group G defined by algebraic difference equations with coefficients in a difference field k are certain elements in $G(A \otimes_k A)$ where A is a difference algebra over k .

Among other things, this has the advantage that we are not restricted to certain nice extensions of difference fields (playing the role of Galois extensions in the classical theory) but we can consider arbitrary difference algebras A instead. In particular, the cohomology set $H_\sigma^1(A/k, G)$ classifies G -torsors that have a point in A . Indeed, this works for arbitrary difference rings k , as long as A is faithfully flat over k .

It seems clear that our approach could be adopted to groups defined by algebraic differential equations as well. This would eliminate the need to consider automorphism of the differential closure and would expedite the calculation of examples.

The outline of the article is as follows: After explaining the basic definitions (such as difference algebraic groups and torsors), we introduce the method of faithfully flat descent in difference algebraic geometry. This is a crucial tool for the proof of our main result (Theorem 4.1; the bijection between the cohomology set and the isomorphism classes of torsors). While for the purpose of this paper faithfully flat descent for difference varieties would be sufficient, we develop faithfully flat descent in the more general context of sheaves since we believe this to be important for the applicability of the method to other problems in difference algebraic geometry.

In Section 3 we introduce the cohomology set $H_\sigma^1(A/k, G)$ and we show that it coincides with the usual cohomology of an algebraic group in the case when G is defined by algebraic equations (rather than actual difference equations).

In Section 4 we then prove the main result: The bijection between $H_\sigma^1(A/k, G)$ and the isomorphism classes of G -torsors that are trivial over A . In Section 5 we introduce the cohomology set $H_\sigma^1(k, G)$ and we show that a short exact sequence of groups defined by difference equations gives rise to a long exact sequence of cohomology sets. We also explicitly compute $H_\sigma^1(k, G)$ for some examples.

In Section 6 we use the results of the previous sections to write down explicitly all torsors for a large class of groups.

Finally, in the last section we present an application to the Galois theory of linear differential equations depending on a discrete parameter. We show that if G is the σ -Galois group of a linear differential equation $y' = Ay$ (with respect to some σ -Picard-Vessiot ring), then the isomorphism classes of all σ -Picard-Vessiot rings for $y' = Ay$ are classified by $H_\sigma^1(k, G)$.

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1. Notation and basic definitions

We first recall some basic notions from difference algebra. Standard references are [Lev08] and [Coh65]. A look at the first chapters of [Wib15] might also be helpful.

All rings are assumed to be commutative and with unity. A difference ring, or σ -ring for short,

is a ring k equipped with a ring homomorphism $\sigma: k \rightarrow k$. If the underlying ring of a σ -ring is a field, it is called a σ -field. A morphism of σ -rings is a morphism of rings that commutes with the action of σ . If $k \rightarrow R$ is a morphism of σ -rings, we also say that R is a k - σ -algebra. A morphism of k - σ -algebras is a morphism of k -algebras that is also a morphism of difference rings. The category of k - σ -algebras is denoted by k - σ -**Alg**. A k - σ -subalgebra of a k - σ -algebra is a k -subalgebra that is stable under σ . If R is a k - σ -algebra, and $B \subseteq R$ a subset, the smallest k - σ -subalgebra of R containing B is denoted by $k\{B\}$. A k - σ -algebra R is called *finitely σ -generated* if $R = k\{B\}$ for a finite subset B of R . If R and S are k - σ -algebras, then $R \otimes_k S$ is naturally a k - σ -algebra via $\sigma(r \otimes s) = \sigma(r) \otimes \sigma(s)$. A σ -field extension L/K is an extension of fields such that L is a K - σ -algebra.

An ideal \mathfrak{a} in a σ -ring R is called a σ -ideal if $\sigma(\mathfrak{a}) \subseteq \mathfrak{a}$. In this case R/\mathfrak{a} naturally carries the structure of a σ -ring. The σ -polynomial ring $k\{y_1, \dots, y_n\}$ over a σ -ring k in the σ -variables y_1, \dots, y_n is the polynomial ring over k in the variables $y_1, \dots, y_n, \sigma(y_1), \dots, \sigma(y_n), \sigma^2(y_1), \dots$. We consider $k\{y_1, \dots, y_n\}$ as a k - σ -algebra by extending σ from k to $k\{y_1, \dots, y_n\}$ as suggested by the names of the variables.

A set of σ -polynomials $F \subseteq k\{y_1, \dots, y_n\}$ defines a functor X from k - σ -**Alg** to **Sets** by $X(R) = \{x \in R^n \mid f(x) = 0 \text{ for all } f \in F\}$ for all k - σ -algebras R . Any such functor will be called a k - σ -variety (or a σ -variety over k). A morphism of k - σ -varieties is a morphism of functors.

For a σ -variety X we define

$$\mathbb{I}(X) = \{f \in k\{y_1, \dots, y_n\} \mid f(x) = 0 \text{ for all } x \in X(R) \text{ and } k\text{-}\sigma\text{-algebras } R\}.$$

Then $\mathbb{I}(X)$ is a σ -ideal in $k\{y_1, \dots, y_n\}$ and

$$k\{X\} = k\{y_1, \dots, y_n\}/\mathbb{I}(X)$$

is a finitely σ -generated k - σ -algebra that we call the *coordinate ring* of X . For any k - σ -algebra R we have a map $\text{Hom}(k\{X\}, R) \rightarrow X(R)$ given by sending a morphism of k - σ -algebras $\psi: k\{X\} \rightarrow R$ to $(\psi(\overline{y_1}), \dots, \psi(\overline{y_n}))$. It is easy to see (Cf. [Wib15, Section 1.2.1].) that this defines an isomorphism of functors $\text{Hom}(k\{X\}, -) \cong X$ and that the category of k - σ -varieties is anti-equivalent to the category of finitely generated k - σ -algebras. In particular, a morphism $\phi: X \rightarrow Y$ of difference varieties corresponds to a morphism $\phi^*: k\{Y\} \rightarrow k\{X\}$ of k - σ -algebras. In the sequel we will sometimes identify $X(R)$ with $\text{Hom}(k\{X\}, R)$.

We allow ourselves the freedom of calling any functor from k - σ -**Alg** to **Sets** that is representable by a finitely σ -generated k - σ -algebra a k - σ -variety.

A σ -closed σ -subvariety Y of a σ -variety X , is a subfunctor Y of X defined by a σ -ideal. More precisely, we require that there exists a σ -ideal $\mathfrak{a} \subseteq k\{X\}$ such that

$$\begin{array}{ccc} Y(R) & \hookrightarrow & X(R) \\ \cong \downarrow & & \downarrow \cong \\ \text{Hom}(k\{X\}/\mathfrak{a}, R) & \hookrightarrow & \text{Hom}(k\{X\}, R) \end{array}$$

commutes for every k - σ -algebra R . For example, if $F \subseteq F' \subseteq k\{y_1, \dots, y_n\}$, then the σ -variety defined by F' is a σ -closed σ -subvariety of the σ -variety defined by F .

A morphism $\phi: X \rightarrow Y$ of σ -varieties is called a σ -closed embedding if it induces an isomorphism between X and a σ -closed σ -subvariety of Y . One can show ([Wib15, Lemma 1.2.7]) that $\phi: X \rightarrow Y$ is a σ -closed embedding if and only if $\phi^*: k\{Y\} \rightarrow k\{X\}$ is surjective.

If R is a σ -ring or a k - σ -algebra, we write $R^\#$ for the ring or k -algebra obtained from R

by forgetting σ . If \mathcal{X} is an affine scheme of finite type over k , then we can define a functor X from $k\text{-}\sigma\text{-}\mathbf{Alg}$ to \mathbf{Sets} by $X(R) = \mathcal{X}(R^\sharp)$ for any $k\text{-}\sigma$ -algebra R . It is easy to see that X is a $k\text{-}\sigma$ -variety: If $\mathcal{X} = \text{Spec}(k[y_1, \dots, y_n]/\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq k[y_1, \dots, y_n]$, then X is the σ -variety defined by $\mathfrak{a} \subseteq k\{y_1, \dots, y_n\}$.

If X is any functor from $k\text{-}\sigma\text{-}\mathbf{Alg}$ to \mathbf{Sets} and A a $k\text{-}\sigma$ -algebra, we denote by X_A the base change functor from $A\text{-}\sigma\text{-}\mathbf{Alg}$ to \mathbf{Sets} given by $X_A(R) = X(R)$ where the $A\text{-}\sigma$ -algebra R is interpreted as a $k\text{-}\sigma$ -algebra via $k \rightarrow A \rightarrow R$. If X is a $k\text{-}\sigma$ -variety X_A is an $A\text{-}\sigma$ -variety (represented by $k\{X\} \otimes_k A$).

The category of $k\text{-}\sigma$ -varieties has products, indeed $k\{X \times Y\} = k\{X\} \otimes_k k\{Y\}$. A σ -algebraic group over k is a group object in the category of $k\text{-}\sigma$ -varieties. A morphism $\phi: G \rightarrow H$ of σ -algebraic groups is a morphism of σ -varieties that is compatible with the group structure, i.e., $\phi_R: G(R) \rightarrow H(R)$ is a morphism of groups for all $k\text{-}\sigma$ -algebras R . A σ -closed subgroup H of a σ -algebraic group G , is a σ -closed σ -subvariety H of G such that $H(R)$ is a subgroup of $G(R)$ for any $k\text{-}\sigma$ -algebra R . If $H(R)$ is a normal subgroup of $G(R)$ for all $k\text{-}\sigma$ -algebras R , then H is called a *normal* σ -closed subgroup.

EXAMPLE 1.1. Let k be a σ -ring and let \mathcal{G} be an algebraic group over k , by which we mean an affine group scheme of finite type over k^\sharp . Then \mathcal{G} can be interpreted as a σ -algebraic group over k by considering the functor G from $k\text{-}\sigma\text{-}\mathbf{Alg}$ to \mathbf{Groups} with $G(R) = \mathcal{G}(R^\sharp)$ for every $k\text{-}\sigma$ -algebra R .

DEFINITION 1.2. Let k be a σ -ring and G a σ -algebraic group over k . A G -torsor is a non-empty σ -variety X over k together with a morphism $\mu: G \times X \rightarrow X$ of σ -varieties such that $\mu_R: G(R) \times X(R) \rightarrow X(R)$ defines a group action of $G(R)$ on $X(R)$ and such that $G(R) \times X(R) \rightarrow X(R) \times X(R)$, $(g, x) \mapsto (\mu_R(g, x), x)$ is bijective for every $k\text{-}\sigma$ -algebra R .

A morphism of G -torsors $\phi: X \rightarrow Y$ is a morphism of σ -varieties that is G -equivariant, i.e.,

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ \text{id} \times \phi \downarrow & & \downarrow \phi \\ G \times Y & \longrightarrow & Y \end{array}$$

commutes.

If X is a G -torsor and R is a $k\text{-}\sigma$ -algebra, we will abbreviate $g.x = \mu_R(g, x)$ for $g \in G(R)$, $x \in X(R)$. If S is another $k\text{-}\sigma$ -algebra and if there is an obvious morphism $R \rightarrow S$ (which will be clear from the context) we will use expressions such as $g.x$ for $g \in G(S)$, $x \in X(R)$ to denote $g.x = g.\alpha(x)$ where $\alpha: X(R) \rightarrow X(S)$.

The bijectivity of $G(R) \times X(R) \rightarrow X(R) \times X(R)$ in Definition 1.2 is equivalent to saying that for $x_1, x_2 \in X(R)$ there exists a unique $g \in G(R)$ with $x_2 = g.x_1$. The group itself $X = G$ is a G -torsor under the group multiplication (from the left).

Let A be a $k\text{-}\sigma$ -algebra and let X be a G -torsor over k . Then X_A is a G_A -torsor and we say that X is *trivial over* A if X_A is isomorphic to G_A as a G_A -torsors.

REMARK 1.3. A G -torsor X is trivial over A if and only if $X(A) \neq \emptyset$.

Proof. If X is trivial over A , then $X(A)$ can be identified with $G(A)$ and is thus non-empty (since $1 \in G(A)$). If $X(A)$ is non-empty, let $x \in X(A)$. For any $A\text{-}\sigma$ -algebra R , set $\gamma_R: G(R) \rightarrow X(R)$, $g \mapsto g.x$. This defines an isomorphism of G_A -torsors $\gamma: G_A \rightarrow X_A$. \square

Clearly every G -torsor X becomes trivial over a finitely σ -generated k - σ -algebra A , namely $A = k\{X\}$. However, as the following example illustrates, if k is a σ -field and X a G -torsor, it is in general not true that X becomes trivial over a σ -field extension of k . Thus, even though we are mostly interested in the case where k is a σ -field, it is important to work with k - σ -algebras A rather than σ -field extensions of k .

EXAMPLE 1.4. Let k be a σ -field and let G be the σ -closed subgroup of the multiplicative group \mathbb{G}_m given by

$$G(R) = \{g \in R^\times \mid g^2 = 1, \sigma(g) = g\}$$

for any k - σ -algebra R .

- a) Let X be the k - σ -variety given by $X(R) = \{x \in R \mid x^2 = 1, \sigma(x) = -x\}$ for every k - σ -algebra R . Then X is a G -torsor ($G(R)$ acts on $X(R)$ via multiplication in R) and X is non-trivial over every σ -field extension K/k , since $X(K) = \emptyset$ for all such K .
- b) More generally, if $a, b \in k^\times$ such that $\sigma(a) = ab^2$, then $X_{a,b}(R) = \{x \in R \mid x^2 = a, \sigma(x) = bx\}$ defines a G -torsor $X_{a,b}$. (Note that the condition $\sigma(a) = ab^2$ guarantees that X is not empty.) Based on our study of H^1 we will see later that in fact every G -torsor is isomorphic to $X_{a,b}$ for a suitable choice of $a, b \in k^\times$ (Example 5.3).

EXAMPLE 1.5. Let k be a σ -field and let G be the σ -closed subgroup of the additive group \mathbb{G}_a given by

$$G(R) = \{g \in R \mid \sigma^n(g) + \lambda_{n-1}\sigma^{n-1}(g) + \dots + \lambda_0 g = 0\}$$

for any k - σ -algebra R , where $\lambda_0, \dots, \lambda_{n-1} \in k$. For $a \in k$ let X_a be given by

$$X_a(R) = \{x \in R \mid \sigma^n(x) + \lambda_{n-1}\sigma^{n-1}(x) + \dots + \lambda_0 x = a\}$$

for any k - σ -algebra R . Then X_a is a G -torsor under the action $g.x = g + x$. Based on our study of H^1 we will see later that in fact every G -torsor is isomorphic to some X_a (Example 5.4).

2. Faithfully flat descent for difference varieties

In this section, we introduce faithfully flat descent for difference varieties. This is a crucial tool for proving the main result (Theorem 4.1) in Section 4, but we also expect this technique to be applicable to other problems in difference algebraic geometry. Of course, this is inspired by faithfully flat descent for schemes. (See e.g., [Sta16, Tag 0238]). We first treat the more general case of sheaves in the context of faithfully flat difference algebras.

Throughout this section, k denotes a σ -ring and A denotes a faithfully flat k - σ -algebra, i.e., A is a k - σ -algebra and A is faithfully flat as a k -algebra.

Sheaves are useful when working with quotients by group actions. (See e.g. [DG70, Chapter III] or [Wib15, Section 5.1].) A functor F from k - σ -**Alg** to **Sets** is called a *sheaf* over k if it satisfies the following two conditions:

- a) For every morphism of k - σ -algebras $R \rightarrow S$ such that S is a faithfully flat R -algebra, the sequence

$$F(R) \rightarrow F(S) \rightrightarrows F(S \otimes_R S)$$

is exact (i.e., an equalizer of sets). In particular, $F(R) \rightarrow F(S)$ is injective.

- b) For every finite family (R_i) of k - σ -algebras, the map $F(\prod R_i) \rightarrow \prod F(R_i)$ induced from the projections $\prod R_i \rightarrow R_{i_0}$ is bijective.

It is easy to see that a representable functor is a sheaf (cf. [Wib15, Cor. 5.1.4]). In particular any k - σ -variety is a sheaf. A morphism of sheaves is a morphism of functors.

If F is a functor from A - σ -**Alg** to **Sets** and B is a k - σ -algebra, then we write $B \otimes_k F$ for the base change $B \otimes_k F: (B \otimes_k A)$ - σ -**Alg** \rightarrow **Sets** of F from A to $B \otimes_k A$. That is, for a $B \otimes_k A$ - σ -algebra R we have $(B \otimes_k F)(R) = F(R)$, where R is considered as an A - σ -algebra via $A \rightarrow A \otimes_k B \rightarrow R$. Similarly we define $F \otimes_k B: (A \otimes_k B)$ - σ -**Alg** \rightarrow **Sets**. If F is a sheaf over A , then $B \otimes_k F$ is a sheaf over $B \otimes_k A$.

Let $F: A$ - σ -**Alg** \rightarrow **Sets** be a sheaf over A . A *descent datum on F* (or more precisely, an A/k -descent datum) is an isomorphism $\phi: A \otimes_k F \rightarrow F \otimes_k A$ such that $\phi_{13} = \phi_{12} \circ \phi_{23}$, where

$$\begin{aligned}\phi_{23}: A \otimes_k A \otimes_k F &\rightarrow A \otimes_k F \otimes_k A \\ \phi_{12}: A \otimes_k F \otimes_k A &\rightarrow F \otimes_k A \otimes_k A \\ \phi_{13}: A \otimes_k A \otimes_k F &\rightarrow F \otimes_k A \otimes_k A\end{aligned}$$

denote the canonical extensions of ϕ . The isomorphisms ϕ_{ij} of functors from $A \otimes_k A \otimes_k A$ - σ -**Alg** to **Sets** correspond to the three canonical possibilities of considering $A \otimes_k A \otimes_k A$ as an $A \otimes_k A$ -algebra as determined by the indices ij . For example, ϕ_{23} corresponds to $A \otimes_k A \rightarrow A \otimes_k A \otimes_k A$, $a \otimes b \mapsto 1 \otimes a \otimes b$.

Let $\tilde{F}: A$ - σ -**Alg** \rightarrow **Sets** be another sheaf with descent datum $\tilde{\phi}: A \otimes_k \tilde{F} \rightarrow \tilde{F} \otimes_k A$. A *morphism of sheaves with descent data* is a morphism of functors $\gamma: F \rightarrow \tilde{F}$ such that the following diagram commutes:

$$\begin{array}{ccc} A \otimes_k F & \xrightarrow{\phi} & F \otimes_k A \\ A \otimes \gamma \downarrow & & \downarrow \gamma \otimes A \\ A \otimes_k \tilde{F} & \xrightarrow{\tilde{\phi}} & \tilde{F} \otimes_k A \end{array}$$

In this situation we also say that γ is compatible with the descent data. If $F_0: k$ - σ -**Alg** \rightarrow **Sets** is a sheaf over k , then there is a canonical A/k -descent datum on $(F_0)_A$ corresponding to the fact that the two maps $A \rightrightarrows A \otimes_k A$ agree on k .

THEOREM 2.1. *The base change from k to A defines an equivalence of categories from the category of sheaves over k to the category of sheaves over A with A/k -descent data.*

Proof. Let F be a sheaf over A with descent datum $\phi: A \otimes_k F \rightarrow F \otimes_k A$. We assign to (F, ϕ) a functor $F_0: k$ - σ -**Alg** \rightarrow **Sets** as follows. For every k - σ -algebra R , consider the diagram

$$\begin{array}{ccc} (A \otimes_k F)(R \otimes_k A \otimes_k A) & \xrightarrow{\phi_{R \otimes A \otimes A}} & (F \otimes_k A)(R \otimes_k A \otimes_k A) \\ & \swarrow \psi_1 \quad \searrow \psi_2 & \\ & F(R \otimes_k A) & \end{array} \quad (1)$$

where ψ_1 and ψ_2 are obtained from the morphisms $R \otimes_k A \rightarrow R \otimes_k A \otimes_k A$ that map $r \otimes a$ to $r \otimes 1 \otimes a$, or to $r \otimes a \otimes 1$; and define

$$F_0(R) = \{x \in F(R \otimes_k A) \mid \phi_{R \otimes A \otimes A}(\psi_1(x)) = \psi_2(x)\}.$$

We claim that F_0 is a sheaf over k such that $(F_0)_A$ is isomorphic to F as a sheaf over A with descent datum. Let $R \rightarrow S$ be a faithfully flat morphism of k - σ -algebras. Then $R \otimes_k A \rightarrow S \otimes_k A$

is a faithfully flat morphisms of A - σ -algebras, so

$$F(R \otimes_k A) \rightarrow F(S \otimes_k A) \rightrightarrows F((S \otimes_k A) \otimes_{(R \otimes_k A)} (S \otimes_k A))$$

is exact, since F is a sheaf. Thus the sequence

$$F(R \otimes_k A) \rightarrow F(S \otimes_k A) \rightrightarrows F(S \otimes_R S \otimes_k A)$$

is exact and it follows that

$$F_0(R) \rightarrow F_0(S) \rightrightarrows F_0(S \otimes_R S)$$

is exact. To verify the second property in the definition of a sheaf, let (R_i) be a finite family of k - σ -algebras. Using the compatibility of tensor products and products and the fact that F is a sheaf, diagram (1) (with $\prod R_i$ in place of R) becomes

$$\begin{array}{ccc} \prod (A \otimes_k F)(R_i \otimes_k A \otimes_k A) & \xrightarrow{\quad} & \prod (F \otimes_k A)(R_i \otimes_k A \otimes_k A) \\ & \nwarrow \quad \nearrow & \\ & \prod F(R_i \otimes_k A) & \end{array}$$

This shows that F_0 is a sheaf.

To prove that $(F_0)_A$ is isomorphic to F as a sheaf with descent datum, we show that for every A - σ -algebra S , the bijection $\phi_{S \otimes A}: F(S \otimes_k A) \rightarrow F(S \otimes_k A)$ restricts to a bijection $F_0(S) \rightarrow F(S)$. (Note that $F(S)$ injects into $F(S \otimes_k A)$ since $S \rightarrow S \otimes_k A$ is faithfully flat.) The two canonical morphisms $S \otimes_k A \rightrightarrows S \otimes_k A \otimes_k A$ induce maps

$$\begin{aligned} \beta_1 &: (A \otimes_k F)(S \otimes_k A) \rightarrow (A \otimes_k A \otimes_k F)(S \otimes_k A \otimes_k A) \\ \beta_2 &: (A \otimes_k F)(S \otimes_k A) \rightarrow (A \otimes_k F \otimes_k A)(S \otimes_k A \otimes_k A) \\ \gamma_1 &: (F \otimes_k A)(S \otimes_k A) \rightarrow (F \otimes_k A \otimes_k A)(S \otimes_k A \otimes_k A) \\ \gamma_2 &: (F \otimes_k A)(S \otimes_k A) \rightarrow (F \otimes_k A \otimes_k A)(S \otimes_k A \otimes_k A) \end{aligned}$$

and we consider the following diagram.

$$\begin{array}{ccc} (A \otimes_k F)(S \otimes_k A) & \xrightleftharpoons[\phi_{23}^{-1} \circ \beta_2]{\beta_1} & (A \otimes_k A \otimes_k F)(S \otimes_k A \otimes_k A) \\ \phi_{S \otimes A} \downarrow & & \downarrow \phi_{13} \\ (F \otimes_k A)(S \otimes_k A) & \xrightleftharpoons[\gamma_2]{\gamma_1} & (F \otimes_k A \otimes_k A)(S \otimes_k A \otimes_k A) \end{array}$$

The diagram commutes with the top arrows and the cocycle condition implies that it also commutes with the bottom arrows. As the vertical arrows are isomorphisms, we conclude that $\phi_{S \otimes A}$ maps the equalizer of the upper horizontal arrows to the equalizer of the lower horizontal arrows. The upper equalizer equals $F_0(S)$ and the lower equalizer can be identified with $F(S)$, since F is a sheaf and $S \otimes_k A$ is faithfully flat over S . Hence $(F_0)_A$ is isomorphic to F and the cocycle condition implies that this isomorphism is compatible with the descent data.

We conclude that the base change functor is essentially surjective. On the other hand, if F_0 and G_0 are sheaves over k and if we denote $F = (F_0)_A$ and $G = (G_0)_A$, then it is easy to check that every morphism $F \rightarrow G$ of sheaves with A/k -descent data is the base change of a unique morphism $F_0 \rightarrow G_0$ of sheaves over k . Hence the base change functor is full and faithful and the claim follows. \square

LEMMA 2.2. *Let $F_0: k\text{-}\sigma\text{-}\mathbf{Alg} \rightarrow \mathbf{Sets}$ be a sheaf over k such that $F = (F_0)_A$ is representable, that is, there exists an A - σ -algebra B such that $F \cong \mathrm{Hom}(B, -)$. Then F_0 is representable.*

Proof. Let $\phi: A \otimes_k F \rightarrow F \otimes_k A$ denote the canonical descent datum on $F = (F_0)_A$ and let $\phi^*: B \otimes_k A \rightarrow A \otimes_k B$ be the corresponding isomorphism of $(A \otimes_k A)$ - σ -algebras. We consider the k - σ -algebra

$$B_0 = \{b \in B \mid \phi^*(b \otimes 1) = 1 \otimes b\}.$$

We claim that $\mathrm{Hom}(B_0, -) \cong F_0$. Theorem 2.1 implies that it suffices to show that $\mathrm{Hom}(A \otimes_k B_0, -) \cong \mathrm{Hom}(B, -)$ as sheaves over A with A/k -descent data. Using the $(A \otimes_k A \otimes_k A)$ - σ -isomorphisms

$$\begin{aligned} \phi_{23}^* &= (\phi_{23})^*: A \otimes_k B \otimes_k A \rightarrow A \otimes_k A \otimes_k B \\ \phi_{12}^* &= (\phi_{12})^*: B \otimes_k A \otimes_k A \rightarrow A \otimes_k B \otimes_k A \\ \phi_{13}^* &= (\phi_{13})^*: B \otimes_k A \otimes_k A \rightarrow A \otimes_k A \otimes_k B \end{aligned}$$

the latter statement can be rephrased using morphisms of σ -rings only.

We have to show that the canonical map $A \otimes_k B_0 \rightarrow B$ is an isomorphism compatible with the descent data. If we disregard the action of σ this statement is well known. (See e.g., [Sta16, Tag 0238].) So the claim follows since all the involved maps are compatible with σ . \square

COROLLARY 2.3. *The base change from k to A defines an equivalence of categories from the category of σ -varieties over k to the category of σ -varieties over A with A/k -descent data.*

Proof. As the k - σ -varieties are exactly the sheaves over k that can be represented by a finitely σ -generated k - σ -algebra, by Theorem 2.1 and Lemma 2.2 it suffices to show the following: If B_0 is a k - σ -algebra such that $B = B_0 \otimes_k A$ is finitely σ -generated over A , then B_0 is finitely σ -generated over k .

As B is σ -generated over A by some finite subset T of $B_0 \otimes_k A$, we can modify T if necessary, to assume $T \subseteq B_0$. Then $k\{T\} \subseteq B_0$ and $k\{T\} \otimes_k A = B_0 \otimes_k A$. Hence $k\{T\} = B_0$, since A is faithfully flat over k . \square

If X and Y are A - σ -varieties with A/k -descent data, $\phi: A \otimes_k X \rightarrow X \otimes_k A$ and $\rho: A \otimes_k Y \rightarrow Y \otimes_k A$ then $\phi \times \rho$ defines a descent datum on $X \times Y$ and this is the product in the category of A - σ -varieties with A/k -descent data.

REMARK 2.4. *By Corollary 2.3 a σ -algebraic group G_0 over k corresponds to a group object in the category of σ -varieties over A with descent datum, i.e., to a σ -algebraic group G over A together with an A/k -descent datum such that the multiplication $G \times G \rightarrow G$, the inversion $G \rightarrow G$ and the identity $1 \rightarrow G$ are compatible with the descent data. Similarly, if X is a G -torsor equipped with a descent datum such that the action $G \times X \rightarrow X$ is compatible with the descent data, then X descends to a G_0 -torsor X_0 , since the isomorphism $G \times X \rightarrow X \times X$, $(g, x) \mapsto (g.x, x)$ descends to an isomorphism $G_0 \times X_0 \rightarrow X_0 \times X_0$.*

For later use we record:

LEMMA 2.5. *Let $\phi: X \rightarrow Y$ be a morphism of k - σ -varieties. If the base extension $\phi_A: X_A \rightarrow Y_A$ is a σ -closed embedding, then ϕ is a σ -closed embedding.*

Proof. If $k\{Y\} \otimes_k A \rightarrow k\{X\} \otimes_k A$ is surjective, then $k\{Y\} \rightarrow k\{X\}$ is surjective since A is faithfully flat over k . \square

3. Cohomology

In this section, we introduce the notion of cohomology sets for σ -algebraic groups. The definition is analogous to the cohomology sets for (not necessarily reduced) algebraic groups. (See e.g., [DG70, III, §4, 6.7] or [Mil80, Chapter III, §4].) Throughout this section, let k be a σ -ring and let G be a σ -algebraic group over k .

Let A be a k - σ -algebra. Consider the group homomorphisms

$$\delta_1, \delta_2: G(A) \rightarrow G(A \otimes_k A),$$

where δ_1 is induced by $A \rightarrow A \otimes_k A$, $a \mapsto 1 \otimes a$ and δ_2 is induced by $A \rightarrow A \otimes_k A$, $a \mapsto a \otimes 1$. Similarly, consider the three group homomorphisms

$$\partial_1, \partial_2, \partial_3: G(A \otimes_k A) \rightarrow G(A \otimes_k A \otimes_k A),$$

where ∂_1 is induced by $A \otimes_k A \rightarrow A \otimes_k A \otimes_k A$, $a \otimes b \mapsto 1 \otimes a \otimes b$ etc.

DEFINITION 3.1. *Let A be a k - σ -algebra. The set of 1-cocycles of G with respect to A/k is defined as*

$$Z_\sigma^1(A/k, G) = \{\chi \in G(A \otimes_k A) \mid \partial_2(\chi) = \partial_1(\chi)\partial_3(\chi)\}.$$

If χ is a cocycle and $\alpha \in G(A)$ it is straight forward to check that $\delta_1(\alpha)\chi\delta_2(\alpha)^{-1}$ is a cocycle.

Two cocycles χ_1 and χ_2 are called equivalent, if there is an $\alpha \in G(A)$ such that $\chi_1 = \delta_1(\alpha)\chi_2\delta_2(\alpha)^{-1}$. The 1-cohomology set of G with respect to A/k is defined as the set of equivalence classes of 1-cocycles:

$$H_\sigma^1(A/k, G) = Z_\sigma^1(A/k, G) / \sim.$$

Recall that a pointed set is simply a set with a distinguished element. A morphism of pointed sets is a map of sets that preserves the distinguished element.

We consider $H_\sigma^1(A/k, G)$ as a pointed set, the distinguished element being the equivalence class of $1 \in G(A \otimes_k A)$. If G is abelian, the product of two cocycles is again a cocycle and the multiplication in $G(A \otimes_k A)$ induces a group structure on $H_\sigma^1(A/k, G)$ and we consider $H_\sigma^1(A/k, G)$ as an abelian group.

EXAMPLE 3.2. If A/k is faithfully flat, then $H_\sigma^1(A/k, \mathbb{G}_a) = \{0\}$.

Proof. Here \mathbb{G}_a is the σ -algebraic group given by $\mathbb{G}_a(R) = (R, +)$ for any k - σ -algebra R . The assertion follows directly from the fact that the following sequence is exact

$$0 \longrightarrow k \longrightarrow A \xrightarrow{\delta_2 - \delta_1} A \otimes_k A \xrightarrow{\partial_3 - \partial_2 + \partial_1} A \otimes_k A \otimes_k A \longrightarrow \dots$$

(See for example [DG70, Chapter I, §1, Section 2.7].) □

REMARK 3.3. If $A \rightarrow B$ is a morphism of k - σ -algebras, then there is a natural map of pointed sets $H_\sigma^1(A/k, G) \rightarrow H_\sigma^1(B/k, G)$. We will see later (Lemma 5.1) that this map is injective and does not depend on the choice of the morphism $A \rightarrow B$.

REMARK 3.4. If $G \rightarrow H$ is a morphism of σ -algebraic groups, then there is a natural map of pointed sets $H_\sigma^1(A/k, G) \rightarrow H_\sigma^1(A/k, H)$.

PROPOSITION 3.5. Let \mathcal{G} be an algebraic group over k^\sharp , that is, an affine group scheme of finite type over k and let G be the corresponding σ -algebraic group over k (see Example 1.1). Then

$$H_\sigma^1(A/k, G) = H^1(A^\sharp/k^\sharp, \mathcal{G})$$

for every k - σ -algebra A .

Proof. Let A be a k - σ algebra. Then δ_1, δ_2 induce morphisms $\delta_1^\sharp, \delta_2^\sharp: \mathcal{G}(A^\sharp) \rightarrow \mathcal{G}(A^\sharp \otimes_{k^\sharp} A^\sharp)$ and $\partial_1, \partial_2, \partial_3$ induce morphisms $\partial_1^\sharp, \partial_2^\sharp, \partial_3^\sharp: \mathcal{G}(A^\sharp \otimes_{k^\sharp} A^\sharp) \rightarrow \mathcal{G}(A^\sharp \otimes_{k^\sharp} A^\sharp \otimes_{k^\sharp} A^\sharp)$.

The 1-cohomology set $H^1(A^\sharp/k^\sharp, \mathcal{G})$ is defined as $H^1(A^\sharp/k^\sharp, \mathcal{G}) = Z^1(A^\sharp/k^\sharp, \mathcal{G})/\sim$, where $Z^1(A^\sharp/k^\sharp, \mathcal{G}) = \{\chi \in \mathcal{G}(A^\sharp \otimes_{k^\sharp} A^\sharp) \mid \partial_2^\sharp(\chi) = \partial_1^\sharp(\chi)\partial_3^\sharp(\chi)\}$ and two cocycles $\chi, \tilde{\chi}$ are called equivalent, if there is an $\alpha \in \mathcal{G}(A^\sharp)$ such that $\tilde{\chi} = \delta_1^\sharp(\alpha)\chi\delta_2^\sharp(\alpha)^{-1}$ (see for example [DG70, Chapter III, §4, Section 6.4]). Therefore, $Z^1(A^\sharp/k^\sharp, \mathcal{G}) = Z_\sigma^1(A/k, G)$ and $H^1(A^\sharp/k^\sharp, \mathcal{G}) = H_\sigma^1(A/k, G)$. \square

COROLLARY 3.6. *Let k be a σ -field and let A be a k - σ -algebra. Then $H_\sigma^1(A/k, G)$ is trivial for G defined as in Proposition 3.5 for each of the following algebraic groups \mathcal{G} : $\mathbb{G}_m, \mathbb{G}_a, \mathrm{GL}_n, \mathrm{SL}_n$.*

Proof. Proposition 3.5 implies that it suffices to show that $H^1(A^\sharp/k^\sharp, \mathcal{G})$ is trivial.

Let Γ denote the absolute Galois group of k^\sharp . It is well known that the Galois cohomology sets $H^1(\Gamma, \mathcal{G})$ are trivial for all of the above groups \mathcal{G} . (See e.g [Ser97, II.1.2, III.1.1] or [Mil15, Cor. 27.8 and 27.9].) For smooth algebraic groups (and all our examples are smooth) these Galois cohomology sets classify \mathcal{G} -torsors ([DG70, Chapter III, §5, Section 3.6]). As $H^1(A^\sharp/k^\sharp, \mathcal{G})$ classifies \mathcal{G} -torsors that become trivial over A^\sharp , also $H^1(A^\sharp/k^\sharp, \mathcal{G})$ has to be trivial. \square

EXAMPLE 3.7. Let k be a σ -field and as in Example 1.4 let G be the σ -algebraic subgroup of the multiplicative group \mathbb{G}_m given by

$$G(R) = \{g \in R^\times \mid g^2 = 1, \sigma(g) = g\}$$

for any k - σ -algebra R . We claim that for every k - σ -algebra A ,

$$H_\sigma^1(A/k, G) \cong \{\alpha \in A^\times \mid \alpha^2 \in k, \sigma(\alpha)/\alpha \in k\}/\sim$$

where we write $\alpha \sim \alpha'$ if and only if there is a $\lambda \in k^\times$ with $(\alpha')^2 = \lambda^2 \alpha^2$ and $\sigma(\alpha')/\alpha' = \sigma(\lambda)/\lambda \cdot \sigma(\alpha)/\alpha$. Note that the right hand side is a group under multiplication and we claim to have an isomorphism of abelian groups.

If $\alpha \in A^\times$ satisfies $\alpha^2 \in k$ and $\sigma(\alpha)/\alpha \in k$, then the element $\chi = \alpha^{-1} \otimes \alpha$ is contained in $G(A \otimes_k A)$ and it is a cocycle for G . Moreover, χ is equivalent to the trivial cocycle if and only if there exists a $\lambda \in k^\times$ with $\lambda \cdot \alpha \in G(A)$ or equivalently, if and only if $\alpha \sim 1$. The assignment $\alpha \mapsto \chi$ thus defines an injective group morphism from the right hand side to $H_\sigma^1(A/k, G)$.

Now let $\chi \in Z_\sigma^1(A/k, G) \subseteq (A \otimes_k A)^\times$. Then χ can also be considered as an element in $Z_\sigma^1(A/k, \mathbb{G}_m)$ and thus there is an $\alpha \in A^\times$ with $\alpha^{-1} \otimes \alpha = \chi$ by Corollary 3.6. Moreover, $\chi^2 = 1$ and $\sigma(\chi) = \chi$ imply $\alpha^2 \in k$ and $\sigma(\alpha)/\alpha \in k$ and we obtain the desired bijection.

EXAMPLE 3.8. Let k be a σ -field and $\mathcal{L}(y) = \sigma^n(y) + \lambda_{n-1}\sigma^{n-1}(y) + \dots + \lambda_0 y$ a linear difference equation over k . As in Example 1.5, let G be the difference algebraic subgroup of \mathbb{G}_a defined by \mathcal{L} , i.e.,

$$G(R) = \{g \in R \mid \mathcal{L}(g) = 0\} \subseteq \mathbb{G}_a(R)$$

for any k - σ -algebra R . Let A be a k - σ -algebra. We claim that

$$H_\sigma^1(A/k, G) \cong \{\alpha \in A \mid \mathcal{L}(\alpha) \in k\}/\sim,$$

as abelian groups, where $\alpha \sim \alpha'$ if and only if $\mathcal{L}(\alpha' - \alpha) \in \mathcal{L}(k)$.

If $\alpha \in A$ satisfies $\mathcal{L}(\alpha) \in k$, then $\chi = 1 \otimes \alpha - \alpha \otimes 1 \in G(A \otimes_k A)$ is a cocycle for G . Moreover, χ is equivalent to the trivial cocycle if and only if there exists a $\lambda \in k$ with $\alpha - \lambda \in G(A)$ and this is equivalent to $\alpha \sim 0$. The assignment $\alpha \mapsto \chi$ thus defines an injective group morphism from the right hand side to $H_\sigma^1(A/k, G)$.

Now let $\chi \in Z_\sigma^1(A/k, G)$ be a cocycle. Since χ is trivial as a cocycle for \mathbb{G}_a (Example 3.2), χ is of the form $\chi = 1 \otimes \alpha - \alpha \otimes 1$ for some $\alpha \in \mathbb{G}_a(A) = A$ and the claim follows.

PROPOSITION 3.9. *Let G and H be σ -algebraic groups over k . Then there is a canonical bijective map $H_\sigma^1(A/k, G \times H) \rightarrow H_\sigma^1(A/k, G) \times H_\sigma^1(A/k, H)$.*

Proof. The projections from $G \times H$ to G and H restrict to maps from $Z_\sigma^1(A/k, G \times H)$ to $Z_\sigma^1(A/k, G)$ and $Z_\sigma^1(A/k, H)$ which induce maps from $H_\sigma^1(A/k, G \times H)$ to $H_\sigma^1(A/k, G)$ and $H_\sigma^1(A/k, H)$. The resulting map $H_\sigma^1(A/k, G \times H) \rightarrow H_\sigma^1(A/k, G) \times H_\sigma^1(A/k, H)$ is bijective. \square

EXAMPLE 3.10. Combining Corollary 3.6 with Proposition 3.9, we obtain: If k is a σ -field then $H_\sigma^1(A/k, \mathbb{G}_a^n)$ and $H_\sigma^1(A/k, \mathbb{G}_m^n)$ are trivial for every k - σ -algebra A .

4. Torsors and cocycles

In this section, we prove our main result which asserts that torsors of σ -algebraic groups can be classified by the corresponding cohomology sets. Throughout this section, let k be a σ -ring, G a σ -algebraic group over k and A a faithfully flat k - σ -algebra.

Let $\Sigma_{A/k}(G)$ be the set of all k -isomorphism classes of G -torsors that are trivial over A . We consider $\Sigma_{A/k}(G)$ as a pointed set, the distinguished element being the class of the trivial G -torsors.

THEOREM 4.1. *There is an isomorphism of pointed sets between $\Sigma_{A/k}(G)$ and $H_\sigma^1(A/k, G)$.*

Proof. First step: Construct a map $\Xi: \Sigma_{A/k}(G) \rightarrow H_\sigma^1(A/k, G)$.

Let X be a G -torsor that is trivial over A . Then X defines a 1-cocycle as follows. Fix an element $x \in X(A)$ (see Remark 1.3). Consider the two natural maps $f_1, f_2: X(A) \rightarrow X(A \otimes_k A)$. More precisely, f_1 is induced by $A \rightarrow A \otimes_k A$, $a \mapsto 1 \otimes a$ and f_2 is induced by $A \rightarrow A \otimes_k A$, $a \mapsto a \otimes 1$. Let $\chi \in G(A \otimes_k A)$ be the unique element with $f_1(x) = \chi \cdot f_2(x)$. We claim that $\partial_2(\chi) = \partial_1(\chi) \partial_3(\chi)$. For the sake of convenience, we denote the three maps $X(A \otimes_k A) \rightarrow X(A \otimes_k A \otimes_k A)$ also by $\partial_1, \partial_2, \partial_3$ and similarly the three maps $(G \times X)(A \otimes_k A) \rightarrow (G \times X)(A \otimes_k A \otimes_k A)$. Note that $\partial_3 \circ f_2 = \partial_2 \circ f_2$, $\partial_2 \circ f_1 = \partial_1 \circ f_1$, and $\partial_3 \circ f_1 = \partial_1 \circ f_2$ on $X(A)$. We compute

$$\begin{aligned} \partial_2(\chi) \cdot \partial_2(f_2(x)) &= \partial_2(\chi \cdot f_2(x)) = \partial_2(f_1(x)) = \partial_1(f_1(x)) = \partial_1(\chi \cdot f_2(x)) \\ &= \partial_1(\chi) \cdot \partial_1(f_2(x)) = \partial_1(\chi) \cdot \partial_3(f_1(x)) = \partial_1(\chi) \cdot \partial_3(\chi \cdot f_2(x)) \\ &= \partial_1(\chi) \cdot (\partial_3(\chi) \cdot \partial_3(f_2(x))) = (\partial_1(\chi) \partial_3(\chi)) \cdot \partial_3(f_2(x)) \\ &= (\partial_1(\chi) \partial_3(\chi)) \cdot \partial_2(f_2(x)) \end{aligned}$$

and the claim follows.

Now we show that another choice of a rational point $y \in X(A)$ yields a cocycle equivalent to χ . Let $\alpha \in G(A)$ be the unique element with $y = \alpha \cdot x$. Let $\tilde{\chi} \in G(A \otimes_k A)$ be the unique element with $f_1(y) = \tilde{\chi} \cdot f_2(y)$. Then

$$\begin{aligned} (\delta_1(\alpha) \chi) \cdot f_2(x) &= \delta_1(\alpha) \cdot f_1(x) = f_1(\alpha \cdot x) = f_1(y) = \tilde{\chi} \cdot f_2(y) = \tilde{\chi} \cdot f_2(\alpha \cdot x) \\ &= \tilde{\chi} \cdot (\delta_2(\alpha) \cdot f_2(x)) = (\tilde{\chi} \delta_2(\alpha)) \cdot f_2(x) \end{aligned}$$

and thus $\delta_1(\alpha) \chi = \tilde{\chi} \delta_2(\alpha)$, so $\chi \sim \tilde{\chi}$.

Let Y be another G -torsor that is trivial over A and such that there is an isomorphism $\gamma: X \rightarrow Y$ of G -torsors over k . Let $g_1, g_2: Y(A) \rightarrow Y(A \otimes_k A)$ be the analogs of f_1, f_2 . Then $\chi \cdot f_2(x) = f_1(x)$ implies $\chi \cdot \gamma_{A \otimes A}(f_2(x)) = \gamma_{A \otimes A}(f_1(x))$, since γ is G -equivariant. Using $\gamma_{A \otimes A}(f_i(x)) =$

$g_i(\gamma_A(x))$, we conclude that the rational point $\gamma_A(x) \in Y(A)$ defines precisely the same cocycle χ as $x \in X(A)$.

Hence $X \mapsto \chi$ defines a map $\Xi: \Sigma_{A/k}(G) \rightarrow H_\sigma^1(A/k, G)$.

Second step: Construct a map $\Upsilon: H_\sigma^1(A/k, G) \rightarrow \Sigma_{A/k}(G)$.

Let $\chi \in G(A \otimes_k A)$ be a 1-cocycle of G with respect to A/k . Define $Y = G_A$ and $\phi: A \otimes_k Y \rightarrow Y \otimes_k A$ by

$$G(S) \rightarrow G(S), \quad g \mapsto g \cdot \chi$$

for every $(A \otimes_k A)$ - σ -algebra S . Then ϕ_{23} is given by multiplication with $\partial_1(\chi)$ from the right, ϕ_{12} is given by multiplication with $\partial_3(\chi)$ from the right and ϕ_{13} is given by multiplication with $\partial_2(\chi)$ from the right. Thus the cocycle condition $\partial_2(\chi) = \partial_1(\chi)\partial_3(\chi)$ implies $\phi_{13} = \phi_{12} \circ \phi_{23}$, and so ϕ is an A/k -descent datum on Y . We consider Y as a G_A -torsor via $\mu: G_A \times Y \rightarrow Y$ given by $G(R) \times G(R) \rightarrow G(R)$, $(g, y) \mapsto gy$ for every A - σ -algebra R . Let $\rho: A \otimes_k G_A \rightarrow G_A \otimes_k A$ be the canonical descent datum on G_A , so that (G_A, ρ) descends to G , i.e., $\rho(g) = g$ for every $(A \otimes_k A)$ - σ -algebra R and every $g \in G(R)$. Then the following diagram commutes

$$\begin{array}{ccc} A \otimes_k (G_A \times Y) & \xrightarrow{\rho \times \phi} & (G_A \times Y) \otimes_k A \\ A \otimes \mu \downarrow & & \downarrow \mu \otimes A \\ A \otimes_k Y & \xrightarrow{\phi} & Y \otimes_k A \end{array}$$

and thus μ is compatible with the descent data. Hence Y descends to a G -torsor X (Remark 2.4). Clearly, X_A is isomorphic to G_A as a G_A -torsor and so X is trivial over A .

Let $\tilde{\chi} \in Z_\sigma^1(A/k, G)$ be a cocycle that is equivalent to χ , i.e., there exists an $\alpha \in G(A)$ such that $\chi = \delta_1(\alpha)\tilde{\chi}\delta_2(\alpha)^{-1}$. Define $\tilde{Y} = G_A$ with descent datum $\tilde{\phi}: A \otimes_k \tilde{Y} \rightarrow \tilde{Y} \otimes_k A$ given by multiplication with $\tilde{\chi}$ from the right. Then \tilde{Y} descends to a G -torsor \tilde{X} over k and we claim that X and \tilde{X} are isomorphic as G -torsors. By Corollary 2.3 it suffices to construct an isomorphism $\gamma: Y \rightarrow \tilde{Y}$ of G_A -torsors that is compatible with the descent data $\phi, \tilde{\phi}$. Define $\gamma: Y \rightarrow \tilde{Y}$ by

$$G_A(R) \rightarrow G_A(R), \quad g \mapsto g \cdot \alpha$$

for every A - σ -algebra R . Then γ is an isomorphism of G_A -torsors. Note that $A \otimes \gamma: A \otimes_k Y \rightarrow A \otimes_k \tilde{Y}$ is given by multiplication with $\delta_1(\alpha)$ from the right and $\gamma \otimes A: Y \otimes_k A \rightarrow \tilde{Y} \otimes_k A$ is given by multiplication with $\delta_2(\alpha)$ from the right. The condition $\chi = \delta_1(\alpha)\tilde{\chi}\delta_2(\alpha)^{-1}$ thus implies that the diagram

$$\begin{array}{ccc} A \otimes_k Y & \xrightarrow{\phi} & Y \otimes_k A \\ A \otimes \gamma \downarrow & & \downarrow \gamma \otimes A \\ A \otimes_k \tilde{Y} & \xrightarrow{\tilde{\phi}} & \tilde{Y} \otimes_k A \end{array}$$

commutes, so γ is compatible with the descent data $\phi, \tilde{\phi}$.

Hence $\chi \mapsto X$ defines a map $\Upsilon: H_\sigma^1(A/k, G) \rightarrow \Sigma_{A/k}(G)$.

Third step: Show that Υ is the inverse of Ξ .

Let \tilde{X} be a G -torsor over k that is trivial over A . Fix an arbitrary element $\tilde{x} \in \tilde{X}(A)$. Let $\chi \in G(A \otimes_k A)$ be the unique element with $f_1(\tilde{x}) = \chi \cdot f_2(\tilde{x})$, where f_1, f_2 are defined as in the first step. Now define $Y = G_A$ with A/k -descent datum $\phi: A \otimes_k Y \rightarrow Y \otimes_k A$ given by

multiplication with χ from the right. We consider Y as a G_A -torsor via multiplication from the left. By the previous step, Y descends to a G -torsor X over k . We claim that X and \tilde{X} are isomorphic as G -torsors.

Let $\tilde{\phi}$ denote the canonical descent datum on \tilde{X}_A . By Corollary 2.3 it suffices to construct an isomorphism $\gamma: Y \rightarrow \tilde{X}_A$ of G_A -torsors that is compatible with the descent data $\phi, \tilde{\phi}$. Define $\gamma: Y \rightarrow \tilde{X}_A$ by $G(R) \rightarrow \tilde{X}(R)$, $g \mapsto g.\tilde{x}$ for every A - σ -algebra R . Clearly, γ is an isomorphism of G_A -torsors. Note that $A \otimes \gamma: A \otimes_k Y \rightarrow A \otimes_k \tilde{X}_A$ is given by $G(S) \rightarrow \tilde{X}(S)$, $g \mapsto g.f_1(\tilde{x})$ for every $(A \otimes_k A)$ - σ -algebra S and similarly $\gamma \otimes A: Y \otimes_k A \rightarrow \tilde{X}_A \otimes_k A$ is given by $G(S) \rightarrow \tilde{X}(S)$, $g \mapsto g.f_2(\tilde{x})$ for every $(A \otimes_k A)$ - σ -algebra S . The condition $f_1(\tilde{x}) = \chi.f_2(\tilde{x})$ thus implies that the diagram

$$\begin{array}{ccc} A \otimes_k Y & \xrightarrow{\phi} & Y \otimes_k A \\ A \otimes \gamma \downarrow & & \downarrow \gamma \otimes A \\ A \otimes_k \tilde{X}_A & \xrightarrow{\tilde{\phi}} & \tilde{X}_A \otimes_k A \end{array}$$

commutes. Therefore, \tilde{X} and X are isomorphic as G -torsors and thus $\Upsilon \circ \Xi = \text{id}_{\Sigma_{A/k}(G)}$.

Conversely, let us start with a cocycle $\chi \in Z^1(A/k, G)$ and set $Y = G_A$ with A/k -descent datum $\phi: A \otimes_k Y \rightarrow Y \otimes_k A$ given by multiplication with χ from the right. Consider Y as a G_A -torsor via multiplication from the left. The second step implies that Y descends to a G -torsor X . We claim that $x := \chi^{-1} \in G(A \otimes_k A) = Y(A \otimes_k A)$ is contained in $X(A)$ and that $f_1(x) = \chi.f_2(x)$, where $f_1, f_2: X(A) \rightarrow X(A \otimes_k A)$ are defined as in the first step. Consider the diagram

$$\begin{array}{ccc} G(A \otimes_k A \otimes_k A) & \xrightarrow{\phi_{A \otimes A \otimes A}} & G(A \otimes_k A \otimes_k A) \\ & \searrow \partial_2 & \nearrow \partial_3 \\ & G(A \otimes_k A) & \end{array}$$

where we consider $A \otimes_k A \otimes_k A$ as an $(A \otimes_k A)$ -algebra via $a \otimes b \mapsto 1 \otimes a \otimes b$. In particular, $\phi_{A \otimes A \otimes A}$ is given by multiplication from the right with $\partial_1(\chi)$. Recall that

$$X(A) = \{z \in G(A \otimes_k A) \mid \phi_{A \otimes A \otimes A}(\partial_2(z)) = \partial_3(z)\}$$

(compare with the proof of Theorem 2.1). We compute

$$\phi_{A \otimes A \otimes A}(\partial_2(x)) = \partial_2(x)\partial_1(\chi) = \partial_2(\chi)^{-1}\partial_1(\chi) = \partial_3(\chi)^{-1} = \partial_3(x).$$

Hence $x = \chi^{-1}$ is contained in $X(A)$.

The action of G on X is defined via restriction of the G_A -action on Y . Thus for every k - σ -algebra R , $G(R)$ acts on $X(R) \subseteq G(R \otimes_k A)$ via multiplication inside $G(R \otimes_k A)$. In particular, $g.z = \partial_3(g) \cdot z$ for every $g \in G(A \otimes_k A)$ and every $z \in X(A \otimes_k A) \subseteq G(A \otimes_k A \otimes_k A)$. Thus $\chi.f_2(x) = \partial_3(\chi).f_2(x)$ with multiplication in $G(A \otimes_k A \otimes_k A)$. Note that $f_1, f_2: X(A) \rightarrow X(A \otimes_k A)$ are the restrictions of $\partial_1, \partial_2: G(A \otimes_k A) \rightarrow G(A \otimes_k A \otimes_k A)$ from $G(A \otimes_k A)$ to $X(A) \subseteq G(A \otimes_k A)$. Thus

$$f_1(x) = \partial_1(x) = \partial_1(\chi)^{-1} = \partial_3(\chi)\partial_2(\chi^{-1}) = \partial_3(\chi)f_2(x) = \chi.f_2(x)$$

as claimed. We conclude that $\Xi \circ \Upsilon = \text{id}$. \square

In the following example we show how Theorem 4.1 can be used to determine all torsors for the σ -closed subgroups of \mathbb{G}_m^n .

EXAMPLE 4.2. Let k be a σ -field and let G be a σ -closed subgroup of \mathbb{G}_m^n . An element $f \in k\{\mathbb{G}_m^n\} = k\{y_1, y_1^{-1}, y_2, y_2^{-1}, \dots, y_n, y_n^{-1}\}$ is called a *multiplicative function* if

$$f = y^{\alpha_0} \sigma(y^{\alpha_1}) \dots \sigma^l(y^{\alpha_l}) \text{ where } \alpha_1, \dots, \alpha_l \in \mathbb{Z}^n$$

and $y^\beta = y_1^{\beta_1} y_2^{\beta_2} \dots y_n^{\beta_n}$ for $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$. In [DVHW14, Lemma A.40] it is shown that G can be defined by using a set F of multiplicative functions and it follows from [Wib15, Theorem 2.3.1] that we may assume F to be finite. So let us fix a finite set $F = \{f_1, \dots, f_m\}$ of multiplicative functions such that

$$G(R) = \{g \in (R^\times)^n \mid f_1(g) = 1, \dots, f_m(g) = 1\}$$

for any k - σ -algebra R . We will show that every G -torsor \tilde{X} is isomorphic to a G -torsor X of the form

$$X(R) = \{x \in (R^\times)^n \mid f_1(x) = a_1, \dots, f_m(x) = a_m\} \quad (2)$$

for any k - σ -algebra R , where $a_1, \dots, a_m \in k^\times$.

Let A be a k - σ -algebra such that \tilde{X} is trivial over A and let $\chi \in G(A \otimes_k A)$ be a cocycle for G corresponding to \tilde{X} (as constructed in the proof of Theorem 4.1). Since χ is trivial when considered as a cocycle for \mathbb{G}_m^n (Example 3.10) there exists a $g \in \mathbb{G}_m^n(A)$ such that $\chi = \delta_1(g)\delta_2(g)^{-1}$. Since $\chi \in G(A \otimes_k A)$ we have $f_i(\chi) = 1$ for $i = 1, \dots, m$. But

$$f_i(\chi) = f_i(\delta_1(g)\delta_2(g)^{-1}) = f_i(g) \otimes (f_i(g))^{-1} \in A \otimes_k A.$$

Thus $a_i = f_i(g) \in k$ for $i = 1, \dots, m$.

Now let X be the G -torsor as defined in Equation (2) above. (Note that X is not empty as $g \in X(A)$.) We claim that X and \tilde{X} are isomorphic. Indeed, the cocycle corresponding to X (constructed from $g \in X(A)$ as in the proof of Theorem 4.1) is χ . Thus X and \tilde{X} are isomorphic by Theorem 4.1.

Let \mathcal{G} be an algebraic group over k and let \mathcal{X} be a \mathcal{G} -torsor. Then clearly the corresponding k - σ -variety X (i.e., $X(R) = \mathcal{X}(R^\#)$ for every k - σ -algebra R) is a G -torsor for the corresponding σ -algebraic group G . The following corollary provides a converse.

COROLLARY 4.3. *Let \mathcal{G} be an algebraic group over a σ -field k and let G be the corresponding σ -algebraic group. Then:*

- a) *Every G -torsor is isomorphic to a G -torsor X , corresponding to a \mathcal{G} -torsor \mathcal{X} .*
- b) *Two G -torsors X and Y as in a) are isomorphic as (difference) G -torsors if and only if \mathcal{X} and \mathcal{Y} are isomorphic as (algebraic) \mathcal{G} -torsors.*

Proof. Since every G -torsor is trivial over some k - σ -algebra this is clear from Proposition 3.5 and Theorem 4.1. \square

The following lemma will be needed in Section 7.

LEMMA 4.4. *Let k be a σ -field and let G be a σ -closed subgroup of GL_n . Then every G -torsor is isomorphic (as a G -torsor) to a σ -closed σ -subvariety X of GL_n , where the action of G on X is given by matrix multiplication.*

Proof. Let X' be a G -torsor and let A be a k - σ -algebra such that X' is trivial over A . Let $\chi \in G(A \otimes_k A)$ be a cocycle corresponding to X' (as constructed in the proof of Theorem 4.1). As in the proof, we use χ to define a descent datum on G_A . Since χ can also be interpreted as a cocycle

for GL_n , it also defines a descent datum on $(\mathrm{GL}_n)_A$. The σ -closed embedding $G_A \rightarrow (\mathrm{GL}_n)_A$ is G_A -equivariant and compatible with the descent data. By Lemma 2.5 it therefore descends to a G -equivariant σ -closed embedding $X' \rightarrow X''$ where X'' is some GL_n -torsor. Since every GL_n -torsor is trivial (Corollary 3.6 and Theorem 4.1) X'' is isomorphic to GL_n with left multiplication. Composing $X' \rightarrow X''$ with this isomorphism yields the desired embedding of X' into GL_n . \square

5. $H_\sigma^1(k, G)$ and the cohomology sequence

In this section k is a σ -field and G is a σ -algebraic group over k . Note that because k is a field any k - σ -algebra A is faithfully flat and therefore the results of the previous sections apply to all k - σ -algebras.

LEMMA 5.1. *Let $A \rightarrow B$ be a morphism of k - σ -algebras. Then the diagram*

$$\begin{array}{ccc} \Sigma_{A/k}(G) & \xrightarrow{\sim} & H_\sigma^1(A/k, G) \\ \downarrow & & \downarrow \\ \Sigma_{B/k}(G) & \xrightarrow{\sim} & H_\sigma^1(B/k, G) \end{array}$$

commutes, where the horizontal arrows are the isomorphisms from Theorem 4.1, the left vertical arrow is the natural inclusion and the right vertical arrow is the canonical map $H_\sigma^1(A/k, G) \rightarrow H_\sigma^1(B/k, G)$. In particular, the latter map is injective and does not depend on the choice of the morphism $A \rightarrow B$.

Proof. Let X be a G -torsor that is trivial over A , so there exists an $x \in X(A)$. Let us fix the following notation: we consider the canonical maps $s: X(A) \rightarrow X(B)$, $r: X(A \otimes_k A) \rightarrow X(B \otimes_k B)$ and $u: G(A \otimes_k A) \rightarrow G(B \otimes_k B)$ induced from $A \rightarrow B$ and the canonical maps $f_1, f_2: X(A) \rightarrow X(A \otimes_k A)$, $g_1, g_2: X(B) \rightarrow X(B \otimes_k B)$.

Then $s(x) \in X(B)$, hence X is trivial over B . Thus there is indeed an inclusion $\Sigma_{A/k}(G) \subseteq \Sigma_{B/k}(G)$. Let $\chi \in G(A \otimes_k A)$ be the unique element with $f_1(x) = \chi \cdot f_2(x)$. By the proof of Theorem 4.1, χ is a cocycle and its equivalence class is the image of the isomorphism class of X under the upper horizontal arrow. The image of the equivalence class of χ under the right vertical arrow is represented by $u(\chi)$. It remains to show that the equivalence class represented by $u(\chi)$ corresponds to the isomorphism class of X under the lower horizontal arrow. Note that $r(f_1(x)) = r(\chi \cdot f_2(x)) = u(\chi) \cdot r(f_2(x))$. As $r \circ f_1 = g_1 \circ s$ and $r \circ f_2 = g_2 \circ s$, we conclude $g_1(s(x)) = u(\chi) \cdot g_2(s(x))$ and the claim follows. \square

Let I denote the set of isomorphism classes of k - σ -algebras. We define a preorder on I as follows: If A and B are k - σ -algebras representing equivalence classes $[A]$ and $[B]$ then $[A] \leq [B]$ if there exists a morphism of k - σ -algebras $A \rightarrow B$. Since $[A], [B] \leq [A \otimes_k B]$ we see that I is a directed set. If $[A] \leq [B]$, according to Lemma 5.1, we have a canonical (injective) morphism of pointed sets $d_{A,B}: H_\sigma^1(A/k, G) \rightarrow H_\sigma^1(B/k, G)$. If $[A] \leq [B]$ and $[B] \leq [C]$ it is clear from Lemma 5.1 that $d_{A,C} = d_{B,C} \circ d_{A,B}$. We thus have a directed system of pointed sets and we can define $H_\sigma^1(k, G)$ as the direct limit

$$H_\sigma^1(k, G) = \varinjlim H_\sigma^1(A/k, G).$$

In general, $H_\sigma^1(k, G)$ is just a pointed set, but if G is abelian, the maps $d_{A,B}$ are morphisms of abelian groups and so $H_\sigma^1(k, G)$ is also an abelian group. Since every G -torsor X becomes trivial over some k - σ -algebra A (e.g., $A = k\{X\}$) we obtain from Theorem 4.1:

THEOREM 5.2. *The cohomology set $H_\sigma^1(k, G)$ is in bijection with the isomorphism classes of G -torsors.* \square

If $\phi: G \rightarrow H$ is a morphism of σ -algebraic groups and $A \rightarrow B$ a morphism of k - σ -algebras, then

$$\begin{array}{ccc} H_\sigma^1(A/k, G) & \longrightarrow & H_\sigma^1(A/k, H) \\ \downarrow & & \downarrow \\ H_\sigma^1(B/k, G) & \longrightarrow & H_\sigma^1(B/k, H) \end{array}$$

commutes and we obtain a map of pointed sets $H_\sigma^1(k, G) \rightarrow H_\sigma^1(k, H)$.

EXAMPLE 5.3. As in Examples 1.4 and 3.7, let G be the σ -algebraic subgroup of \mathbb{G}_m given by

$$G(R) = \{g \in R^\times \mid g^2 = 1, \sigma(g) = g\}$$

for any k - σ -algebra R . Let M be the set of all pairs $(a, b) \in k^\times \times k^\times$ such that $\sigma(a) = ab^2$. Define an equivalence relation on M by $(a, b) \sim (a', b')$ if there exists a $\lambda \in k^\times$ such that $a' = \lambda^2 a$ and $b' = \frac{\sigma(\lambda)}{\lambda} b$. Note that M is an abelian group under multiplication and also $\mathbb{H} = M / \sim$ is an abelian group. We claim that

- (1) $H_\sigma^1(k, G) \cong \mathbb{H}$.
- (2) Every G -torsor is isomorphic to $X_{a,b}$ for some $a, b \in k^\times$ with $\sigma(a) = ab^2$, where $X_{a,b}$ is given by $X_{a,b}(R) = \{x \in R \mid x^2 = a, \sigma(x) = bx\}$ for any k - σ -algebra R .
- (3) Two G -torsors $X_{a,b}$ and $X_{a',b'}$ are isomorphic if and only if $(a, b) \sim (a', b')$.

Let A be a k - σ -algebra. We saw in Example 3.7 that

$$H_\sigma^1(A/k, G) \cong \{\alpha \in A^\times \mid \alpha^2 \in k, \sigma(\alpha)/\alpha \in k\} / \sim$$

where $\alpha \sim \alpha'$ if there is a $\lambda \in k^\times$ with $(\alpha')^2 = \lambda^2 \alpha^2$ and $\sigma(\alpha')/\alpha' = \sigma(\lambda)/\lambda \cdot \sigma(\alpha)/\alpha$. Setting $a = \alpha^2$ and $b = \sigma(\alpha)/\alpha$ yields a morphism of abelian groups $H_\sigma^1(A/k, G) \rightarrow \mathbb{H}$. If $[A] \leq [B]$, then

$$\begin{array}{ccc} H_\sigma^1(A/k, G) & \longrightarrow & H_\sigma^1(B/k, G) \\ & \searrow & \swarrow \\ & \mathbb{H} & \end{array}$$

commutes and we obtain a morphism $H_\sigma^1(k, G) \rightarrow \mathbb{H}$.

Conversely, let $(a, b) \in k^\times \times k^\times$ with $\sigma(a) = ab^2$. Then we can define a k - σ -algebra structure on $A = k[y]/(y^2 - a)$ by setting $\sigma(y) = by$. If $(a, b) \sim (a', b')$ then $A \cong A'$ via $y \mapsto \frac{1}{\lambda} y'$. The image α of y in A satisfies $\alpha^2 = a \in k$ and $\sigma(\alpha)/\alpha = b \in k$. Then $\chi = \delta_1(\alpha)\delta_2(\alpha)^{-1} \in Z_\sigma^1(A/k, G)$ and since $\lambda\alpha \mapsto \alpha'$ under $A \cong A'$ we see that the image of χ in $H_\sigma^1(k, G)$ only depends on the equivalence class of (a, b) . So we have a map $\mathbb{H} \rightarrow H_\sigma^1(k, G)$ and it is clear that these two maps are inverse to each other, so (1) follows.

To prove (2), let X be a G -torsor and fix a k - σ -algebra A such that X is trivial over A . We already saw in Example 3.7, that a cocycle $\chi \in Z_\sigma^1(A/k, G)$ corresponding to X is of the form $\chi = \alpha^{-1} \otimes \alpha$ for some $\alpha \in A^\times$ with $\alpha^2, \sigma(\alpha)/\alpha \in k^\times$. Hence the torsor $X_{a,b}$ with $a = \alpha^2$ and $b = \sigma(\alpha)/\alpha$ is isomorphic to X , since X and $X_{a,b}$ define the same equivalence class in $H_\sigma^1(A/k, G)$. For (3), assume $(a, b) \sim (a', b')$ and let $\lambda \in k^\times$ be such that $a' = \lambda^2 a$ and $b' = \frac{\sigma(\lambda)}{\lambda} b$.

Then $x \mapsto \lambda x$ defines an isomorphism of G -torsors $X_{a,b} \cong X_{a',b'}$. Conversely, if (a, b) and (a', b') are not equivalent, then $X_{a,b}$ and $X_{a',b'}$ correspond to different elements in $\mathbb{H} \cong H_\sigma^1(k, G)$ and therefore are not isomorphic.

EXAMPLE 5.4. Let k be a σ -field and $\mathcal{L}(y) = \sigma^n(y) + \lambda_{n-1}\sigma^{n-1}(y) + \dots + \lambda_0 y$ a linear difference equation over k . As in Examples 1.5 and 3.8, we consider the σ -closed subgroup G of \mathbb{G}_a given by

$$G(R) = \{g \in R \mid \mathcal{L}(g) = 0\} \subseteq \mathbb{G}_a(R)$$

for any k - σ -algebra R . Note that if k has characteristic zero, then any σ -closed subgroup of \mathbb{G}_a is of this form ([DVHW15, Cor. A.3]).

We define an equivalence relation on k by $a \sim a'$ if there exists an element $b \in k$ such that $a' - a = \mathcal{L}(b)$. Then $\mathbb{H} = k / \sim = k / \mathcal{L}(k)$ is an abelian group under addition and we claim that

- (1) $H_\sigma^1(k, G) \cong \mathbb{H}$. In particular, $H_\sigma^1(k, G)$ is trivial, if and only if $\mathcal{L}: k \rightarrow k$ is surjective.
- (2) Every G -torsor is isomorphic to X_a for some $a \in k$, where X_a is defined by $X_a(R) = \{x \in R \mid \mathcal{L}(x) = a\}$ for any k - σ -algebra R .
- (3) Two G -torsors X_a and $X_{a'}$ are isomorphic if and only if $a \sim a'$.

Let A be a k - σ -algebra. We saw in Example 3.8 that

$$H_\sigma^1(A/k, G) \cong \{\alpha \in A \mid \mathcal{L}(\alpha) \in k\} / \sim,$$

where $\alpha \sim \alpha'$ if $\mathcal{L}(\alpha' - \alpha) \in \mathcal{L}(k)$. Setting $a = \mathcal{L}(\alpha)$ defines a morphism of abelian groups $H_\sigma^1(A/k, G) \rightarrow \mathbb{H}$ and this induces a morphism of abelian groups $H_\sigma^1(k, G) \rightarrow \mathbb{H}$.

Conversely, for $a \in k$ let A be a k - σ -algebra such that there exists an $\alpha \in A$ with $\mathcal{L}(\alpha) = a$. (For example, we may take $A = k[y_1, \dots, y_n]$ with $\sigma(y_1) = y_2, \sigma(y_2) = y_3, \dots, \sigma(y_n) = -\lambda_{n-1}y_n - \dots - \lambda_0 y_1$ and $\alpha = y_1 \in A$.) Then $\chi = 1 \otimes \alpha - \alpha \otimes 1$ is a cocycle for G . The image of χ in $H_\sigma^1(k, G)$ only depends on the equivalence class of a , and the induced mapping $\mathbb{H} \rightarrow H_\sigma^1(k, G)$ is the inverse to the mapping constructed previously. Hence (1) follows.

For (2), let X be a G -torsor and A a k - σ -algebra such that X is trivial over A . We already saw in Example 3.8 that a cocycle $\chi \in Z_\sigma^1(A/k, G)$ corresponding to X is of the form $\chi = 1 \otimes \alpha - \alpha \otimes 1$ for some $\alpha \in A$ with $\mathcal{L}(\alpha) \in k$. Set $a = \mathcal{L}(\alpha)$. Since X_a and X correspond to the same element in $H_\sigma^1(A/k, G)$, we see that X_a and X are isomorphic.

Finally, let $a, a' \in k$ such that there exists an element $b \in k$ with $a' - a = \mathcal{L}(b)$. Then $x \mapsto x + b$ defines an isomorphism $X_a \rightarrow X_{a'}$ of G -torsors. Conversely, if a and a' are not equivalent, then the torsors X_a and $X_{a'}$ correspond to different elements in $\mathbb{H} \cong H_\sigma^1(k, G)$ and therefore are not isomorphic. Hence (3) follows.

From Corollary 3.6 and Example 3.10 we obtain:

EXAMPLE 5.5. The cohomology set $H_\sigma^1(k, G)$ is trivial for all σ -algebraic groups G corresponding to the algebraic groups \mathbb{G}_m^n , \mathbb{G}_a^n , GL_n and SL_n .

To establish the long exact cohomology sequence we need to know a few things about quotients of σ -algebraic groups. For a morphism $\phi: G \rightarrow H$ of σ -algebraic groups we set $\ker(\phi)(R) = \ker(\phi_R)$ for any k - σ -algebra R .

If N is a normal σ -closed subgroup of G there exists a morphism of σ -algebraic groups $\pi: G \rightarrow G/N$ satisfying the following universal property: $N \subseteq \ker(\pi)$ and for any morphism of

σ -algebraic groups $\phi: G \rightarrow H$ such that $N \subseteq \ker(\phi)$ there exists a unique morphism $G/N \rightarrow H$ such that

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/N \\ & \searrow \phi & \swarrow \text{dotted} \\ & H & \end{array}$$

commutes. Moreover, $\ker(\pi) = N$. (See [Wib15, Theorem 3.2.3].) The morphism $\pi: G \rightarrow G/N$ is in a certain sense surjective (see [Wib15, Theorem 3.3.6] for more details), but it is not surjective in the naive sense that $\pi_R: G(R) \rightarrow (G/N)(R)$ is surjective for all k - σ -algebras R . However, what is true, is that for every k - σ -algebra R and every $h \in (G/N)(R)$ there exists an R - σ -algebra S and a $g \in G(S)$ such that $\pi(g) \in (G/N)(S)$ equals the image of h under $(G/N)(R) \rightarrow (G/N)(S)$.

Recall that a sequence of pointed sets $M' \rightarrow M \rightarrow M''$ is exact (at M) if the inverse image of the distinguished element of M'' equals the image of M' in M .

PROPOSITION 5.6. *Let N be a normal σ -closed subgroup of G . Then there exists a morphism of pointed sets $\delta: (G/N)(k) \rightarrow H_\sigma^1(k, N)$ such that the sequence*

$$1 \rightarrow N(k) \rightarrow G(k) \rightarrow (G/N)(k) \xrightarrow{\delta} H_\sigma^1(k, N) \rightarrow H_\sigma^1(k, G) \rightarrow H_\sigma^1(k, G/N)$$

of pointed sets is exact.

Proof. To define δ , let $x \in (G/N)(k)$. Then there exists a k - σ -algebra A and an element $g \in G(A)$ such that $\pi(g) = x$ (more precisely, π maps g to the image of x in $(G/N)(A)$). Define $\chi = \delta_1(g)\delta_2(g)^{-1} \in G(A \otimes_k A)$. As x is k -rational, $\pi(\chi) = 1$. Hence $\chi \in N(A \otimes_k A)$ (by [Wib14, Thm. 7.3]) and χ is a cocycle for N . We would like to define $\delta(x) \in H_\sigma^1(k, G)$ as the image of the equivalence class of χ but we first need to show that this is independent of the choice of A and g . Let B be another k - σ -algebra such that there exists an element $h \in G(B)$ with $\pi(h) = x$ and define $\mu = \delta_1(h)\delta_2(h)^{-1} \in N(B \otimes_k B)$.

Let C be a k - σ -algebra with $[C] \geq [A], [B]$ and fix morphisms $A \rightarrow C$ and $B \rightarrow C$. We will show that χ and μ have the same image in $H_\sigma^1(C/k, N)$. (So that we have a well-defined map δ .) We consider the canonical maps $w: G(A) \rightarrow G(C)$, $t: G(B) \rightarrow G(C)$, $u: G(A \otimes_k A) \rightarrow G(C \otimes_k C)$ and $v: G(B \otimes_k B) \rightarrow G(C \otimes_k C)$. It suffices to show that $u(\chi)$ and $v(\mu)$ are equivalent. Since x is k -rational, $\pi(w(g)) = \pi(t(h))$ and so there exists an $n \in N(C)$ such that $w(g) = nt(h)$. Thus $u(\chi) = \delta_1(w(g))\delta_2(w(g))^{-1} = \delta_1(n)\delta_1(t(h))\delta_2(t(h))^{-1}\delta_2(n)^{-1}$ is equivalent to $\delta_1(t(h))\delta_2(t(h))^{-1} = v(\mu)$.

The exactness at $N(k)$ is trivial and the exactness at $G(k)$ follows from $\ker(\pi) = N$. To prove exactness at $(G/N)(k)$, let $x \in (G/N)(k)$ be such that $\delta(x)$ is trivial. Fix a k - σ -algebra A such that there is a $g \in G(A)$ with $\pi(g) = x$. As $\delta(x)$ is trivial, there exists an $n \in N(A)$ with $\delta_1(g)\delta_2(g)^{-1} = \delta_1(n)\delta_2(n)^{-1}$. Thus $n^{-1}g$ is contained in $G(k)$ and $\pi(n^{-1}g) = x$.

To prove exactness at $H_\sigma^1(k, N)$, let A be a k - σ -algebra and let $\chi \in Z_\sigma^1(A/k, N)$ be a cocycle whose class is trivial in $Z_\sigma^1(A/k, G)$. Thus there exists a $g \in G(A)$ with $\chi = \delta_1(g)\delta_2(g)^{-1}$. Define $x = \pi(g) \in (G/N)(A)$. As $\chi \in N(A \otimes_k A)$, $\pi(\chi) = 1$ and so $\delta_1(\pi(g)) = \delta_2(\pi(g))$. Thus $x \in (G/N)(k)$ and the claim follows.

To prove exactness at $H_\sigma^1(k, G)$ let A be a k - σ -algebra and let $\chi \in Z_\sigma^1(A/k, G)$ be a cocycle whose class maps to the trivial class under $H_\sigma^1(k, G) \rightarrow H_\sigma^1(k, G/N)$. So there exists an $x \in (G/N)(A)$ such that $\pi(\chi) = \delta_1(x)\delta_2(x)^{-1}$. Let B be an A - σ -algebra such that there exists a $g \in G(B)$ with $\pi(g) = x$. Consider the canonical map $u: G(A \otimes_k A) \rightarrow G(B \otimes_k B)$. Then $\pi(u(\chi)) = \pi(\delta_1(g)\delta_2(g)^{-1})$, so $\delta_1(g)^{-1}u(\chi)\delta_2(g)$ is contained in $N(B \otimes_k B)$ and thus defines a

cocycle in $Z_\sigma^1(B/k, N)$. The image of $\delta_1(g)^{-1}u(\chi)\delta_2(g)$ in $H_\sigma^1(k, N)$ maps to the image of χ in $H_\sigma^1(k, G)$. \square

EXAMPLE 5.7. Let G be a σ -algebraic group and let $d \geq 1$. Note that $\sigma^d: k \rightarrow k$ is a morphism of σ -fields. Thus the base extension $\sigma^d G$ of G via $\sigma^d: k \rightarrow k$ is a σ -algebraic group over k . Moreover, we have a morphism of σ -algebraic groups $\sigma^d: G \rightarrow \sigma^d G$ defined as follows: For a k - σ -algebra R let ${}_{\sigma^d}R$ denote the k - σ -algebra whose underlying difference ring is R with k -algebra structure given by $k \xrightarrow{\sigma^d} k \rightarrow R$. Then $\sigma^d G(R) = G({}_{\sigma^d}R)$ and $\sigma^d: R \rightarrow {}_{\sigma^d}R$ is a morphism of k - σ -algebras. We thus have a morphism of groups $\sigma^d: G(R) \rightarrow \sigma^d G(R)$. Let $N = \{g \in G \mid \sigma^d(g) = 1\}$ be the kernel of $\sigma^d: G \rightarrow \sigma^d G$. The morphism $\sigma^d: G \rightarrow \sigma^d G$ is surjective if and only if G is absolutely σ -reduced, i.e., $\sigma: k\{G\} \otimes_k K \rightarrow k\{G\} \otimes_k K$ is injective for all σ -field extensions K of k , or equivalently, $k\{G\}$ is a σ -separable k - σ -algebra. (See [Wib15, Section 6.3.4] and [TW15, Section 1.1] for more details.) Let us assume that G is absolutely σ -reduced so that $\sigma^d: G \rightarrow \sigma^d G$ is the quotient of G mod N .

We will show that $H_\sigma^1(k, N)$ is trivial if $H_\sigma^1(k, G)$ is trivial and $\sigma: k \rightarrow k$ is an automorphism. According to Proposition 5.6 we have an exact sequence

$$G(k) \xrightarrow{\sigma^d} \sigma^d G(k) \xrightarrow{\delta} H_\sigma^1(k, N) \rightarrow H_\sigma^1(k, G).$$

But since $\sigma: k \rightarrow k$ is bijective, also $G(k) \xrightarrow{\sigma^d} \sigma^d G(k)$ is bijective and it follows that $H_\sigma^1(k, N)$ is trivial.

If $\sigma: k \rightarrow k$ is not an automorphism, $H_\sigma^1(k, N)$ need not be trivial. For example, if $G = \mathrm{GL}_n$, then for a matrix $a \in \mathrm{GL}_n(k)$ with $a \notin \mathrm{GL}_n(\sigma(k))$, the N -torsor X given by $X(R) = \{x \in \mathrm{GL}_n(R) \mid \sigma^d(x) = a\}$ for any k - σ -algebra R is non-trivial.

6. Classification results

In this section we use Theorem 4.1 to determine all torsors for a large class of σ -algebraic groups.

Throughout this section k is a σ -field. Let \mathcal{X} be an affine scheme of finite type over k^\sharp . To simplify the notation we will drop the \sharp in the sequel, e.g., for a k - σ -algebra R we will write $\mathcal{X}(R)$ instead of $\mathcal{X}(R^\sharp)$.

For $d \geq 1$ let $\sigma^d \mathcal{X}$ be the affine scheme of finite type over k obtained from \mathcal{X} by base extension via $\sigma^d: k \rightarrow k$. Similarly, if $\varphi: \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of affine schemes of finite type over k , then we let $\sigma^d \varphi: \sigma^d \mathcal{X} \rightarrow \sigma^d \mathcal{Y}$ be the morphism obtained from φ by base extension via $\sigma^d: k \rightarrow k$. More explicitly, $\sigma^d \mathcal{X}(S) = \mathcal{X}({}_{\sigma^d}S)$ for every k -algebra S , where ${}_{\sigma^d}S$ denotes the ring S considered as a k -algebra via $k \xrightarrow{\sigma^d} k \rightarrow S$. Note that $(\sigma^d \varphi)_S = \varphi({}_{\sigma^d}S): \mathcal{X}({}_{\sigma^d}S) \rightarrow \mathcal{Y}({}_{\sigma^d}S)$.

If \mathcal{G} is an algebraic group over k (i.e., an affine group scheme of finite type over k) and \mathcal{X} is a \mathcal{G} -torsor, then $\sigma^d \mathcal{G}$ is an algebraic group over k and $\sigma^d \mathcal{X}$ is a $\sigma^d \mathcal{G}$ -torsor via the base extension $\sigma^d \mathcal{G} \times \sigma^d \mathcal{X} \rightarrow \sigma^d \mathcal{X}$ of $\mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X}$.

For every k - σ -algebra R the map $\sigma^d: R \rightarrow {}_{\sigma^d}R$ is a morphism of k -algebras and therefore induces a map $\sigma^d: \mathcal{X}(R) \rightarrow \sigma^d \mathcal{X}(R)$.

REMARK 6.1. If $\theta: \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of affine schemes of finite type over k , then the following equation holds for all k - σ -algebras R and all $x \in \mathcal{X}(R)$: $\sigma^d(\theta(x)) = \sigma^d \theta(\sigma^d(x))$.

THEOREM 6.2. Let \mathcal{G} be an algebraic group over k and for some fixed $d \geq 1$ let $\psi: \mathcal{G} \rightarrow \sigma^d \mathcal{G}$ be

a morphism of algebraic groups. We consider the σ -algebraic group G with

$$G(R) = \{g \in \mathcal{G}(R) \mid \sigma^d(g) = \psi(g)\}$$

for all k - σ -algebras R . The following holds:

- a) Let \mathcal{X} be a \mathcal{G} -torsor and let $\varphi: \mathcal{X} \rightarrow \sigma^d \mathcal{X}$ be a morphism such that $\varphi(g.x) = \psi(g).\varphi(x)$ for all k -algebras S and all $g \in \mathcal{G}(S), x \in \mathcal{X}(S)$. Then

$$Y(R) = \{x \in \mathcal{X}(R) \mid \sigma^d(x) = \varphi(x)\}$$

for all k - σ -algebras R defines a G -torsor Y .

- b) Every G -torsor X is isomorphic to a G -torsor Y as defined in a).

- c) Two such G -torsors Y and \tilde{Y} given by $Y(R) = \{x \in \mathcal{X}(R) \mid \sigma^d(x) = \varphi(x)\}$ and $\tilde{Y}(R) = \{x \in \tilde{\mathcal{X}}(R) \mid \sigma^d(x) = \tilde{\varphi}(x)\}$ are isomorphic if and only if there exists an isomorphism of \mathcal{G} -torsors $\Theta: \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ such that $\sigma^d \Theta \circ \varphi = \tilde{\varphi} \circ \Theta$.

Proof. For a), it is easy to check that the action $\mathcal{G} \times \mathcal{X} \rightarrow \mathcal{X}$ restricts to $G \times Y \rightarrow Y$ and that Y is a G -torsor.

To prove b), we fix a k - σ -algebra A with $X(A) \neq \emptyset$ and let $[\chi] \in H_\sigma^1(A/k, G)$ be the equivalence class that corresponds to the isomorphism class of X (use Theorem 4.1). As $G(R) \subseteq \mathcal{G}(R)$ for all k - σ -algebras R , there is a canonical map $H_\sigma^1(A/k, G) \rightarrow H^1(A/k, \mathcal{G})$ and the image of $[\chi]$ thus corresponds to a \mathcal{G} -torsor \mathcal{X} that is also trivial over A (see for example [DG70, Chapter III, §4, 6.5]). Recall that $\chi \in G(A \otimes_k A)$ and that up to an isomorphism of torsors,

$$\begin{aligned} \mathcal{X}(S) &= \{g \in \mathcal{G}(S \otimes_k A) \mid \psi_2(g) = \psi_1(g) \cdot \chi\} \\ \sigma^d \mathcal{X}(S) &= \mathcal{X}(\sigma^d S) = \{g \in \mathcal{G}(\sigma^d S \otimes_k A) \mid \phi_2(g) = \phi_1(g) \cdot \chi\} \end{aligned}$$

for all k -algebras S , where ψ_1, ψ_2 denote the two canonical maps $\mathcal{G}(S \otimes_k A) \rightarrow \mathcal{G}(S \otimes_k A \otimes_k A)$ and $\phi_1, \phi_2: \mathcal{G}(\sigma^d S \otimes_k A) \rightarrow \mathcal{G}(\sigma^d S \otimes_k A \otimes_k A)$. As \mathcal{X} corresponds to the cocycle χ , there exists an $x \in \mathcal{X}(A)$ with $f_1(x) = \chi.f_2(x)$, where $f_1, f_2: \mathcal{X}(A) \rightarrow \mathcal{X}(A \otimes_k A)$. Let $y = \sigma^d(x) \in \sigma^d \mathcal{X}(A)$ and compute $g_1(y) = \sigma^d(\chi).g_2(y)$, where $g_1, g_2: \sigma^d \mathcal{X}(A) \rightarrow \sigma^d \mathcal{X}(A \otimes_k A)$. As $\sigma^d \mathcal{X}$ is a $\sigma^d \mathcal{G}$ -torsor, we conclude that the isomorphism class of $\sigma^d \mathcal{X}$ corresponds to $[\sigma^d(\chi)] \in H^1(A/k, \sigma^d \mathcal{G})$. Thus $\sigma^d \mathcal{X}$ is isomorphic to the $\sigma^d \mathcal{G}$ -torsor \mathcal{Y} defined by

$$\mathcal{Y}(S) = \{g \in \mathcal{G}(\sigma^d S \otimes_k \sigma^d A) \mid \gamma_2(g) = \gamma_1(g) \cdot \sigma^d(\chi)\},$$

where $\gamma_1, \gamma_2: \mathcal{G}(\sigma^d S \otimes_k \sigma^d A) \rightarrow \mathcal{G}(\sigma^d S \otimes_k \sigma^d A \otimes_k \sigma^d A)$. We can also construct an explicit isomorphism $\gamma: \sigma^d \mathcal{X} \rightarrow \mathcal{Y}$: Let S be a k -algebra. Then $s \otimes a \rightarrow s \otimes \sigma^d(a)$ defines a morphism of k -algebras $\sigma^d S \otimes_k A \rightarrow \sigma^d S \otimes_k \sigma^d A$ which induces a group homomorphism $\gamma: \mathcal{G}(\sigma^d S \otimes_k A) \rightarrow \mathcal{G}(\sigma^d S \otimes_k \sigma^d A)$. It is easy to check that γ restricts to a map $\gamma_S: \sigma^d \mathcal{X}(S) \rightarrow \mathcal{Y}(S)$ and that these maps $(\gamma_S)_S$ define a morphism of functors $\gamma: \sigma^d \mathcal{X} \rightarrow \mathcal{Y}$. Moreover, γ is a morphism of $\sigma^d \mathcal{G}$ -torsors and it follows that γ is in fact an isomorphism.

We continue with the construction of a morphism $\varphi: \mathcal{X} \rightarrow \sigma^d \mathcal{X}$. It is easy to check that $\psi_{S \otimes A}: \mathcal{G}(S \otimes_k A) \rightarrow \sigma^d \mathcal{G}(S \otimes_k A) = \mathcal{G}(\sigma^d S \otimes_k \sigma^d A)$ restricts to a map $\rho_S: \mathcal{X}(S) \rightarrow \mathcal{Y}(S)$ for every k -algebra S and that this defines a morphism of functors $\rho: \mathcal{X} \rightarrow \mathcal{Y}$. Let S be a k -algebra and let $g \in \mathcal{G}(S)$ and $x \in \mathcal{X}(S) \subseteq \mathcal{G}(S \otimes_k A)$. Recall that $g.x \in \mathcal{X}(S)$ is obtained from the multiplication of g with x inside $\mathcal{G}(S \otimes_k A)$ (compare with the second step in the proof of Theorem 4.1). Thus $\rho(g.x) = \psi(g.x) = \psi(g) \cdot \psi(x) = \psi(g) \cdot \rho(x)$. Let $\varphi: \mathcal{X} \rightarrow \sigma^d \mathcal{X}$ denote the composition $\varphi = \gamma^{-1} \circ \rho$. Then $\varphi(g.x) = \psi(g).\varphi(x)$ for all k -algebras S and all $g \in \mathcal{G}(S), x \in \mathcal{X}(S)$. The isomorphism class of X corresponds to $[\chi] \in H_\sigma^1(A/k, G)$ and thus X is isomorphic to the torsor Y with

$Y(R) = \{g \in G(R \otimes_k A) \mid \epsilon_2(g) = \epsilon_1(g) \cdot \chi\}$ for all k - σ -algebras R , where ϵ_1, ϵ_2 denote the canonical maps $G(R \otimes_k A) \rightarrow G(R \otimes_k A \otimes_k A)$. We conclude

$$\begin{aligned} Y(R) &= \{g \in G(R \otimes_k A) \mid \epsilon_2(g) = \epsilon_1(g) \cdot \chi\} \\ &= \{g \in \mathcal{G}(R \otimes_k A) \mid \sigma^d(g) = \psi(g) \text{ and } \psi_2(g) = \psi_1(g) \cdot \chi\} \\ &= \{g \in \mathcal{X}(R) \mid \sigma^d(g) = \varphi(g)\}, \end{aligned}$$

where $\psi_1, \psi_2: \mathcal{G}(R \otimes_k A) \rightarrow \mathcal{G}(R \otimes_k A \otimes_k A)$. The claim follows.

To prove part c), let us first assume that there exists an isomorphism of \mathcal{G} -torsors $\Theta: \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ with $\sigma^d \Theta \circ \varphi = \tilde{\varphi} \circ \Theta$. We claim that Θ restricts to an isomorphism $\theta: Y \rightarrow \tilde{Y}$. Let R be a k - σ -algebra. Then for $x \in Y(R)$, we use Remark 6.1 to obtain $\sigma^d(\Theta(x)) = \sigma^d \Theta(\sigma^d(x)) = \sigma^d \Theta(\varphi(x)) = \tilde{\varphi}(\Theta(x))$ and thus $\Theta(x) \in \tilde{Y}(R)$. Conversely, let $y \in \tilde{Y}(R)$ and let $x \in \mathcal{X}(R)$ be the unique element with $\Theta(x) = y$. Then $\sigma^d(y) = \tilde{\varphi}(y)$ implies $\sigma^d \Theta(\sigma^d(x)) = \sigma^d \Theta(\varphi(x))$ and as $\sigma^d \Theta$ is an isomorphism we conclude $\sigma^d(x) = \varphi(x)$. Thus $x \in Y(R)$ and the claim follows.

Conversely, assume that Y and \tilde{Y} are isomorphic as G -torsors. Fix a k - σ -algebra A such that Y and \tilde{Y} are trivial over A . Let further $\chi \in Z^1_\sigma(A/k, G)$ be a cocycle such that its equivalence class corresponds to the isomorphism class of Y and \tilde{Y} . Then there exist elements $y \in Y(A)$ and $\tilde{y} \in \tilde{Y}(A)$ such that $f_1(y) = \chi \cdot f_2(y)$ and $g_1(\tilde{y}) = \chi \cdot g_2(\tilde{y})$, where $f_1, f_2: Y(A) \rightarrow Y(A \otimes_k A)$ and $g_1, g_2: \tilde{Y}(A) \rightarrow \tilde{Y}(A \otimes_k A)$. Consider χ as a cocycle in $Z^1(A/k, \mathcal{G})$. Since $Y(A) \subseteq \mathcal{X}(A)$ and $\tilde{Y}(A) \subseteq \tilde{\mathcal{X}}(A)$, the isomorphism classes of \mathcal{X} and $\tilde{\mathcal{X}}$ both correspond to the equivalence class $[\chi] \in H^1(A/k, \mathcal{G})$. Thus \mathcal{X} and $\tilde{\mathcal{X}}$ are isomorphic as \mathcal{G} -torsors (see [DG70, Chapter III, §4, 6.5]) and moreover, they are both isomorphic to the \mathcal{G} -torsor \mathcal{X}' with

$$\mathcal{X}'(S) = \{g \in \mathcal{G}(S \otimes_k A) \mid \psi_2(g) = \psi_1(g) \cdot \chi\}$$

for all k -algebras S , where $\psi_1, \psi_2: \mathcal{G}(S \otimes_k A) \rightarrow \mathcal{G}(S \otimes_k A \otimes_k A)$. More explicitly, there are isomorphisms $\vartheta: \mathcal{X}' \rightarrow \mathcal{X}$ and $\tilde{\vartheta}: \mathcal{X}' \rightarrow \tilde{\mathcal{X}}$ obtained from descending the isomorphisms $\mathcal{G}_A \rightarrow \mathcal{X}_A$ and $\mathcal{G}_A \rightarrow \tilde{\mathcal{X}}_A$ that are given by $g \mapsto g \cdot y$ and $g \mapsto g \cdot \tilde{y}$ (compare with the third step of the proof of Theorem 4.1). We claim that $\Theta = \tilde{\vartheta} \circ \vartheta^{-1}$ has the desired property $\sigma^d \Theta \circ \varphi = \tilde{\varphi} \circ \Theta$. It suffices to show that this holds after base extension from k to A , so we claim that

$$\sigma^d \tilde{\vartheta}_A \circ \sigma^d \vartheta_A^{-1} \circ \varphi_A = \tilde{\varphi}_A \circ \tilde{\vartheta}_A \circ \vartheta_A^{-1}.$$

We use $(\sigma^d \vartheta)_A = \sigma^d(\vartheta_A)$ to obtain that $\sigma^d \vartheta_A: \sigma^d \mathcal{G}_A \rightarrow \sigma^d \mathcal{X}_A$ is given by $g \mapsto g \cdot \sigma^d(y)$ and similarly, $\sigma^d \tilde{\vartheta}_A: \sigma^d \mathcal{G}_A \rightarrow \sigma^d \tilde{\mathcal{X}}_A$ is given by $g \mapsto g \cdot \sigma^d(\tilde{y})$. Let S be an A -algebra and $z \in \mathcal{X}(S)$. Then there exists a $g \in \mathcal{G}(S)$ with $z = g \cdot y$. We use $y \in Y(A)$, $\tilde{y} \in \tilde{Y}(A)$ to compute

$$\begin{aligned} \sigma^d \tilde{\vartheta}(\sigma^d \vartheta^{-1}(\varphi(g \cdot y))) &= \sigma^d \tilde{\vartheta}(\sigma^d \vartheta^{-1}(\psi(g) \cdot \varphi(y))) \\ &= \sigma^d \tilde{\vartheta}(\sigma^d \vartheta^{-1}(\psi(g) \cdot \sigma^d(y))) \\ &= \psi(g) \cdot \sigma^d(\tilde{y}) \\ &= \psi(g) \cdot \tilde{\varphi}(\tilde{y}) \\ &= \tilde{\varphi}(g \cdot \tilde{y}) \\ &= \tilde{\varphi}(\tilde{\vartheta}(\vartheta^{-1}(g \cdot y))) \end{aligned}$$

and the claim follows. \square

COROLLARY 6.3. *As in Theorem 6.2 we consider the σ -algebraic group G given by*

$$G(R) = \{g \in \mathcal{G}(R) \mid \sigma^d(g) = \psi(g)\}$$

for all k - σ -algebras R . Let X be a G -torsor and assume¹ that $H^1(k, \mathcal{G}) = \{0\}$. Then there exists an element $a \in {}^{\sigma^d}\mathcal{G}(k)$ such that X is isomorphic to the G -torsor Y given by

$$Y(R) = \{x \in \mathcal{G}(R) \mid \sigma^d(x) = \psi(x)a\}$$

for all k - σ -algebras R .

Proof. According to Theorem 6.2 the G -torsor X is isomorphic to Y given by

$$Y(R) = \{x \in \mathcal{G}(R) \mid \sigma^d(x) = \varphi(x)\}$$

for all k - σ -algebras R , where $\varphi: \mathcal{G} \rightarrow {}^{\sigma^d}\mathcal{G}$ is a morphism of schemes over k such that $\varphi(g.x) = \psi(g).\varphi(x)$ for all k -algebras S and all $g, x \in \mathcal{G}(S)$. So if $a = \varphi(1) \in {}^{\sigma^d}\mathcal{G}(k)$ then $\varphi(x) = \psi(x)a$ for all k - σ -algebras R and $x \in \mathcal{G}(R)$. \square

From Corollary 6.3 we obtain a classification of torsors for (special) unitary groups:

EXAMPLE 6.4. Let $\mathcal{G} = \mathrm{GL}_n$ or $\mathcal{G} = \mathrm{SL}_n$. If

$$G = \{g \in \mathcal{G} \mid \sigma^d(g) = (g^T)^{-1}\},$$

then for every G -torsor X , there exists an element $a \in \mathcal{G}(k)$ such that X is isomorphic to the G -torsor

$$Y = \{g \in \mathcal{G} \mid \sigma^d(g) = (g^T)^{-1}a\}.$$

Similarly, if

$$G = \{g \in \mathcal{G} \mid \sigma^d(g) = g\},$$

then for every G -torsor X , there exists an element $a \in \mathcal{G}(k)$ such that X is isomorphic to the G -torsor

$$Y = \{g \in \mathcal{G} \mid \sigma^d(g) = ga\}.$$

COROLLARY 6.5. Let \mathcal{G} be a linear algebraic group over k and consider the σ -algebraic group G with

$$G(R) = \{g \in \mathcal{G}(R) \mid \sigma^d(g) = 1\}$$

for all k - σ -algebras R . Then for every G -torsor X , there exists a \mathcal{G} -torsor \mathcal{X} and an element $a \in {}^{\sigma^d}\mathcal{X}(k)$, such that X is isomorphic to the G -torsor Y with

$$Y(R) = \{x \in \mathcal{X}(R) \mid \sigma^d(x) = a\}$$

for every k - σ -algebra R .

Proof. If we let $\psi: \mathcal{G} \rightarrow {}^{\sigma^d}\mathcal{G}$ be the morphism given by $g \mapsto 1$, then G is defined as in Theorem 6.2. Let X be a G -torsor. By part b) of this theorem, there exists a \mathcal{G} -torsor \mathcal{X} and a morphism $\varphi: \mathcal{X} \rightarrow {}^{\sigma^d}\mathcal{X}$ with $\varphi(g.x) = \varphi(x)$ (for all $x \in \mathcal{X}(S)$, $g \in \mathcal{G}(S)$ and all k -algebras S) such that X is isomorphic to the G -torsor Y with $Y(R) = \{x \in \mathcal{X}(R) \mid \sigma^d(x) = \varphi(x)\}$ for all k - σ -algebras R . Note that $\varphi_S: \mathcal{X}(S) \rightarrow {}^{\sigma^d}\mathcal{X}(S)$ is a constant map for every k -algebra S , i.e., there exists an element $a_S \in {}^{\sigma^d}\mathcal{X}(S)$ with $\varphi(x) = a_S$ for all $x \in \mathcal{X}(S)$. Since φ is a morphism, a_S maps to $a_{S \otimes_k S}$ under both maps ${}^{\sigma^d}\mathcal{X}(S) \rightarrow {}^{\sigma^d}\mathcal{X}(S \otimes_k S)$. As ${}^{\sigma^d}\mathcal{X}$ is a sheaf, we obtain that $a_S \in {}^{\sigma^d}\mathcal{X}(k)$. Thus there is an element $a \in {}^{\sigma^d}\mathcal{X}(k)$ such that $\varphi_S(x) = a$ for all $x \in \mathcal{X}(S)$ and for every k -algebra S . (In particular, the ${}^{\sigma^d}\mathcal{G}$ -torsor ${}^{\sigma^d}\mathcal{X}$ is trivial.) \square

¹Note that this assumption is satisfied if k is algebraically closed.

COROLLARY 6.6. *Under the assumptions of Corollary 6.5, assume in addition that σ is an automorphism on k . Then every G -torsor is trivial (i.e., $H_\sigma^1(k, G) = \{0\}$).*

Proof. Let X be a G -torsor. By Corollary 6.5, there exists a \mathcal{G} -torsor \mathcal{X} and an element $a \in \sigma^d \mathcal{X}(k)$ such that X is isomorphic to the G -torsor Y with $Y(R) = \{x \in \mathcal{X}(R) \mid \sigma^d(x) = a\}$ for every k - σ -algebra R . Since $\sigma^d: k \rightarrow k$ is a bijection also $\sigma^d: \mathcal{X}(k) \rightarrow \sigma^d \mathcal{X}(k)$ is bijective. This shows that $Y(k) \neq \emptyset$. Thus Y is trivial. \square

Corollary 6.6 is not valid without the assumption that $\sigma: k \rightarrow k$ is surjective. For example, if we take $\mathcal{G} = \mathbb{G}_m$ and $a \in k \setminus \sigma(k)$, then $X = \{x \in \mathbb{G}_m \mid \sigma^d(x) = a\}$ is a non-trivial torsor for $G = \{g \in \mathbb{G}_m \mid \sigma(g) = 1\}$.

7. Isomorphism classes of σ -Picard-Vessiot rings

Difference algebraic groups occur naturally as the Galois groups of linear differential equations depending on a discrete parameter. See [DVHW14] and [DVHW15]. A similar Galois theory also exists for linear difference equations ([OW15]). In this last section we present an application to this Galois theory. We show that $H_\sigma^1(k, G)$ classifies isomorphism classes of σ -Picard-Vessiot rings. This generalizes a result proved in the Tannakian context in [DM82, Theorem 3.2]. (See also [Dyc08, Cor. 3.2].)

Before recalling the basics of the σ -Picard-Vessiot theory we discuss a result that only pertains to G -torsors and will be needed in the proofs later. In the following discussion, let k be a σ -field and let G be a σ -algebraic group over k . Let X be a k - σ -variety equipped with a (left) G -action, i.e., a morphism of σ -varieties $G \times X \rightarrow X$ such that $G(R) \times X(R) \rightarrow X(R)$ is a group action for every k - σ -algebra R . We will define a functorial (left) action of G on $k\{X\}$. For every k - σ -algebra R the group $G(R)$ acts on $k\{X\} \otimes_k R$

$$G(R) \times (k\{X\} \otimes_k R) \rightarrow k\{X\} \otimes_k R, (g, f) \mapsto g(f)$$

as follows. The element $g \in G(R)$ defines an automorphism of X_R by $g: X_R \rightarrow X_R, x \mapsto g^{-1}.x$ and the dual map $k\{X\} \otimes_k R \rightarrow k\{X\} \otimes_k R, f \mapsto g(f)$ is an R - σ -algebra automorphism.

An element $f \in k\{X\}$ is called *G -invariant* if $g(f \otimes 1) = f \otimes 1$ for all $g \in G(R)$ and all k - σ -algebras R . The k - σ -subalgebra of $k\{X\}$ consisting of all G -invariant elements is denoted by $k\{X\}^G$. An ideal $\mathfrak{a} \subseteq k\{X\}$ is called *G -stable* if $g(\mathfrak{a} \otimes_k R) \subseteq \mathfrak{a} \otimes_k R$ for all $g \in G(R)$ and all k - σ -algebras R .

LEMMA 7.1. *If X is a G -torsor, then*

- a) $k\{X\}^G = k$ and
- b) *the only G -stable ideals in $k\{X\}$ are $\{0\}$ and $k\{X\}$.*

Proof. a) Let $f \in k\{X\}^G$. Consider X as a right G -torsor via $X \times G \rightarrow X, (x, g) \mapsto g^{-1}.x$ and let $\rho: k\{X\} \rightarrow k\{X\} \otimes_k k\{G\}$ be the dual map of this action. Let R be a k - σ -algebra and $g \in G(R)$. The action of $g \in G(R) = \text{Hom}(k\{G\}, R)$ on $k\{X\} \otimes_k R$ is the R -linear extension of $k\{X\} \xrightarrow{\rho} k\{X\} \otimes_k k\{G\} \xrightarrow{\text{id} \otimes g} k\{X\} \otimes_k R$. By choosing $R = k\{G\}$ and $g = \text{id} \in G(R) = \text{Hom}(k\{G\}, k\{G\})$, we obtain $\rho(f) = g(f) = f \otimes 1$. Since X is a right G -torsor, the left $k\{X\}$ -linear extension $k\{X\} \otimes_k k\{X\} \rightarrow k\{X\} \otimes_k k\{G\}$ of ρ is an isomorphism. This implies that $f \otimes 1 = 1 \otimes f \in k\{X\} \otimes_k k\{X\}$. Thus $f \in k$.

b) Let $\mathfrak{a} \subseteq k\{X\}$ be a G -stable ideal. As in a), this implies that $\rho(\mathfrak{a}) \subseteq \mathfrak{a} \otimes_k k\{G\}$. Now under the isomorphism $k\{X\} \otimes_k k\{X\} \rightarrow k\{X\} \otimes_k k\{G\}$ the ideal $\mathfrak{a} \otimes_k k\{G\} \subseteq k\{X\} \otimes_k k\{G\}$

corresponds to $\mathfrak{a} \otimes_k k\{X\}$ and the ideal of $k\{X\} \otimes_k k\{G\}$ generated by $\rho(\mathfrak{a})$ corresponds to $k\{X\} \otimes_k \mathfrak{a}$. Thus $k\{X\} \otimes_k \mathfrak{a} \subseteq \mathfrak{a} \otimes_k k\{X\}$. This is only possible if \mathfrak{a} is trivial. \square

Let us now recall the basic definitions and results from [DVHW14]. A δ -ring is a commutative ring R together with a derivation $\delta: R \rightarrow R$. An ideal $\mathfrak{a} \subseteq R$ such that $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$ is called a δ -ideal. If every δ -ideal of R is trivial, R is called δ -simple. The δ -constants of R are $R^\delta = \{r \in R \mid \delta(r) = 0\}$.

A $\delta\sigma$ -ring is a commutative ring R together with a derivation $\delta: R \rightarrow R$ and a ring endomorphism $\sigma: R \rightarrow R$ such that $\delta(\sigma(r)) = \hbar\sigma(\delta(r))$ for all $r \in R$ for some fixed unit $\hbar \in R^\delta$. If R is a field, we speak of a $\delta\sigma$ -field. There are the obvious notions of a morphism of $\delta\sigma$ -rings, of $\delta\sigma$ -algebras etc.

From now on let K denote a $\delta\sigma$ -field of characteristic zero and let $k = K^\delta$ be the σ -field of δ -constants. We consider a linear differential system $\delta(y) = Ay$ with a matrix $A \in K^{n \times n}$.

DEFINITION 7.2. A σ -Picard-Vessiot ring² for $\delta(y) = Ay$ is a K - $\delta\sigma$ -algebra R such that

- there exists a matrix $Y \in \mathrm{GL}_n(R)$ with $\delta(Y) = AY$ and $R = K\{Y, 1/\det(Y)\}$,
- R is δ -simple,
- $R^\delta = k$.

The σ -Galois group of R/K is the σ -algebraic group G over k given by

$$G(S) = \mathrm{Aut}^{\delta\sigma}(R \otimes_k S \mid K \otimes_k S)$$

for any k - σ -algebra S . Here and in the sequel $R \otimes_k S$ is considered as a $\delta\sigma$ -ring with δ being the trivial derivation on S , i.e., $\delta(s) = 0$ for $s \in S$. The choice of $Y \in \mathrm{GL}_n(R)$ determines a σ -closed embedding of G into GL_n . Indeed, for every k - σ -algebra S and $g \in G(S)$ there exists a matrix $\phi_S(g) \in \mathrm{GL}_n(S)$ such that $g(Y) = Y\phi_S(g)$. Then $\phi: G \rightarrow \mathrm{GL}_n$ is a σ -closed embedding. The coordinate ring of G is $k\{G\} = (R \otimes_K R)^\delta$ and the canonical map

$$R \otimes_k k\{G\} \rightarrow R \otimes_K R \tag{3}$$

is an isomorphism. This isomorphism can be reinterpreted by saying that R is the coordinate ring of a right G_K -torsor. Using the isomorphism (3), it then follows as in Lemma 7.1 that:

LEMMA 7.3.

- a) $R^G = K$ and
- b) every G -stable ideal of R is trivial. \square

Now we are prepared to prove the main result of this section:

THEOREM 7.4. Let R be a σ -Picard-Vessiot ring for $\delta(y) = Ay$ with σ -Galois group G . Then the set of isomorphism classes of σ -Picard-Vessiot rings for $\delta(y) = Ay$ is in bijection with $H_\sigma^1(k, G)$.

Proof. By Theorem 5.2 it suffices to establish a bijection between the isomorphism classes of σ -Picard-Vessiot rings for $\delta(y) = Ay$ and the isomorphism classes of G -torsors. Indeed, we will construct an isomorphism of pointed sets such that the equivalence class of R corresponds to the equivalence class of the trivial G -torsor.

²This definition differs from Definition [DVHW14, Definition 1.2] where the condition $R^\delta = k$ is dropped. Definition 7.2 is more convenient for us and [DVHW14, Prop. 1.5] shows that with our definition, σ -Picard-Vessiot rings correspond to σ -Picard-Vessiot extensions.

First step: From a σ -Picard-Vessiot ring to a G -torsor:

Let R' be a σ -Picard-Vessiot ring for $\delta(y) = Ay$. For every k - σ -algebra S we set

$$X(S) = \text{Isom}_{K \otimes_k S}^{\delta\sigma}(R' \otimes_k S, R \otimes_k S). \quad (4)$$

We claim that X is a k - σ -variety, i.e., representable by a finitely σ -generated k - σ -algebra. Set $k\{X\} = (R \otimes_K R')^\delta$. According to Lemma 1.8 in [DVHW14] the k - σ -algebra $k\{X\}$ is finitely σ -generated and the canonical map $R \otimes_k k\{X\} \rightarrow R \otimes_K R'$ is an isomorphism (of $\delta\sigma$ -rings). Moreover, as in [DVHW14, Lemma 2.4] (for the case $R' = R$) one shows that every $K \otimes_k S$ - $\delta\sigma$ -morphism from $R' \otimes_k S$ to $R \otimes_k S$ is an isomorphism. Thus we have the following chain of identifications

$$\begin{aligned} X(S) &= \text{Hom}_{K \otimes_k S}^{\delta\sigma}(R' \otimes_k S, R \otimes_k S) \\ &= \text{Hom}_K^{\delta\sigma}(R', R \otimes_k S) \\ &= \text{Hom}_R^{\delta\sigma}(R \otimes_K R', R \otimes_k S) \\ &= \text{Hom}_R^{\delta\sigma}(R \otimes_k k\{X\}, R \otimes_k S) \\ &= \text{Hom}_k^\sigma(k\{X\}, S), \end{aligned}$$

where the last equality holds because $(R \otimes_k S)^\delta = S$. This shows that X is a k - σ -variety. Now G acts on X by composition: $g.x = g \circ x$ for $g \in G(S)$ and $x \in X(S)$ and clearly X is a (left) G -torsor.

Second step: From a G -torsor to a σ -Picard-Vessiot ring

Let X be a G -torsor. Recall that G acts (functorially from the left) on R and on $k\{X\}$. Thus G also acts (functorially from the left) on $R \otimes_k k\{X\}$. Since G acts via K - $\delta\sigma$ -automorphisms the invariants $R' = (R \otimes_k k\{X\})^G$ are a K - $\delta\sigma$ -algebra. We will show that R' is a σ -Picard-Vessiot ring for $\delta(y) = Ay$.

Let us first show that the canonical map $R \otimes_K R' \rightarrow R \otimes_k k\{X\}$ is injective. For a contradiction, assume it is not injective and let $n \geq 2$ be minimal such that there exist n K -linearly independent elements $r'_1, \dots, r'_n \in R'$ that are R -linearly dependent (when viewed as elements in $R \otimes_k k\{X\}$). Let $\mathfrak{a} \subseteq R$ denote the non-zero ideal of all $r_1 \in R$ such that there exist $r_2, \dots, r_n \in R$ with $r_1 r'_1 + r_2 r'_2 + \dots + r_n r'_n = 0$.

We will show that \mathfrak{a} is G -stable. So let S be a k - σ -algebra and $g \in G(S)$. We have to show that $g(r_1 \otimes 1) \in \mathfrak{a} \otimes_k S$ for all $r_1 \in \mathfrak{a}$. As $g(r'_i \otimes 1) = r'_i \otimes 1$ for $i = 1, \dots, n$ we have $g(r_1 \otimes 1)(r'_1 \otimes 1) + \dots + g(r_n \otimes 1)(r'_n \otimes 1) = 0$.

Let $(f_j)_{j \in J}$ be a k -basis of S with $1 \in J$ and $f_1 = 1$. Since G acts on R , we have $g(r_i \otimes 1) \in R \otimes_k k \otimes_k S \subseteq R \otimes_k k\{X\} \otimes_k S$ for $i = 1, \dots, n$. So we can write

$$g(r_i \otimes 1) = \sum_j a_{ij} \otimes 1 \otimes f_j$$

with $a_{ij} \in R$ for $i = 1, \dots, n$. Then

$$\sum_i \sum_j a_{ij} r'_i \otimes f_j = 0$$

and therefore $\sum_i a_{ij} r'_i = 0$ for all $j \in J$. So $a_{1j} \in \mathfrak{a}$ for all $j \in J$. This shows that $g(r_1 \otimes 1) = \sum_j a_{1j} \otimes f_j \in \mathfrak{a} \otimes_k S$. Thus \mathfrak{a} is G -stable. Since R itself is the only non-zero G -stable ideal in R (Lemma 7.3) we see that $1 \in \mathfrak{a}$. So we may assume that $r_1 = 1$. Then $g(r_1 \otimes 1) = 1 \otimes 1 \otimes 1 = a_{11} \otimes 1 \otimes f_1$, i.e., $a_{11} = 1$ and $a_{1j} = 0$ for $j \neq 1$. As $\sum_i a_{ij} r'_i = 0$, it follows from the minimality

of n that $a_{ij} = 0$ for all $i = 1, \dots, n$ and $j \neq 1$. Thus $g(r_i \otimes 1) = a_{i1} \otimes 1 \otimes 1$ for $i = 1, \dots, n$. But then $\sum a_{i1} r'_i = 0$ and so the minimality of n and the fact that $a_{11} = 1$ implies that $a_{i1} = r_i$ for $i = 1, \dots, n$. Thus $g(r_i \otimes 1) = r_i \otimes 1$ for $i = 1, \dots, n$. Hence $r_i \in R^G = K$ for $i = 1, \dots, n$; a contradiction.

This shows that $R \otimes_K R' \rightarrow R \otimes_k k\{X\}$ is injective. Our next concern is to find a $Y' \in \mathrm{GL}_n(R')$ with $\delta(Y') = AY'$. As explained after Definition 7.2, we can consider G as a σ -closed subgroup of GL_n . So by Lemma 4.4, we can assume without loss of generality that X is a σ -closed σ -subvariety of GL_n and that the (left) action of G on X is given by matrix multiplication. Let $k\{\mathrm{GL}_n\} = k\{T, 1/\det(T)\}$ be the (difference) coordinate ring of GL_n and let Z denote the image of T under the morphism $k\{\mathrm{GL}_n\} \rightarrow k\{X\}$ corresponding to the inclusion $X \subseteq \mathrm{GL}_n$. For any k - σ -algebra S and $g \in G(S)$ we have $g(Z) = g^{-1}Z$ and therefore

$$g(Y \otimes Z) = Yg \otimes g^{-1}Z = Y \otimes Z.$$

Thus $Y' = Y \otimes Z \in \mathrm{GL}_n(R')$. Since the entries of Z are δ -constants it is clear that $\delta(Y') = AY'$. Moreover, as $1 \otimes Z = (Y \otimes 1)^{-1}Y'$, the map $R \otimes_K K\{Y', 1/\det(Y')\} \rightarrow R \otimes_k k\{X\}$ is surjective. Since $R \otimes_K K\{Y', 1/\det(Y')\} \subseteq R \otimes_K R'$ and $R \otimes_K R'$ injects into $R \otimes_k k\{X\}$, this implies that $R' = K\{Y', 1/\det(Y')\}$ and $R \otimes_K R' \cong R \otimes_k k\{X\}$.

Let us next show that R' is δ -simple. Let \mathfrak{a} be a proper differential ideal in R' and let $\tilde{\mathfrak{a}}$ be the ideal generated by \mathfrak{a} in $R \otimes_k k\{X\}$. Since $R \otimes_K \mathfrak{a} \subseteq R \otimes_K R'$ is a proper δ -ideal, also $\tilde{\mathfrak{a}} \subseteq R \otimes_k k\{X\}$ is a proper differential ideal. By [DVHW14, Lemma 2.3] the δ -ideal $\tilde{\mathfrak{a}}$ is generated by $\tilde{\mathfrak{a}} \cap k\{X\}$. Moreover, since the elements of \mathfrak{a} are fixed by the G -action, it is clear that $\tilde{\mathfrak{a}}$ is G -stable, so also $\tilde{\mathfrak{a}} \cap k\{X\} \subseteq k\{X\}$ is G -stable. But according to Lemma 7.1, all G -stable ideals in $k\{X\}$ are trivial. Thus $\tilde{\mathfrak{a}} \cap k\{X\} = \{0\}$ and therefore also $\tilde{\mathfrak{a}}$ and \mathfrak{a} are the zero ideal.

Thus R' is δ -simple. To finish the proof that R' is a σ -Picard-Vessiot ring it suffices to show that $R'^\delta = k$. But as $(R \otimes_k k\{X\})^\delta = k\{X\}$, we see that

$$R'^\delta = k\{X\}^G = k$$

by Lemma 7.1. Thus R' is a σ -Picard-Vessiot ring for $\delta(y) = Ay$.

It remains to see that the above two constructions induce bijections on the isomorphism classes. If we start with a σ -Picard-Vessiot ring R' and define X as in step one, then $R \otimes_k k\{X\} \cong R \otimes_K R'$. We will show that this isomorphism is compatible with the functorial G -action if we let G act on R' trivially: If $Y \in \mathrm{GL}_n(R)$ with $\delta(Y) = AY$ and $Y' \in \mathrm{GL}_n(R')$ with $\delta(Y') = AY'$, then $k\{X\} = k\{Z, 1/\det(Z)\}$ where $Z = Y^{-1} \otimes Y' \in \mathrm{GL}_n(R \otimes_K R')$ ([DVHW14, Lemma 1.8]). Thus we can consider X as a σ -closed σ -subvariety of GL_n . For a k - σ -algebra S , an element $x \in X(S) \subseteq \mathrm{GL}_n(S)$ corresponds to the isomorphism $R' \otimes_k S \rightarrow R \otimes_k S$ determined by $Y' \mapsto Yx$. Moreover, the composition with an element $g \in G(S)$ corresponds to the isomorphism determined by $Y' \mapsto Ygx$. Thus G acts on X by matrix multiplication from the left. So on the coordinate ring $k\{X\}$, the action of G is given by $g(Z) = g^{-1}Z = g^{-1}(Y^{-1} \otimes Y')$. This shows that the isomorphism $R \otimes_k k\{X\} \cong R \otimes_K R'$ is compatible with the G -action if $g(Y') = Y'$. Since $(R \otimes_K R')^G \cong R'$ it follows that also $(R \otimes_k k\{X\})^G \cong R'$.

Conversely, if we start with a G -torsor X and define $R' = (R \otimes_k k\{X\})^G$, then $R \otimes_K R' \rightarrow R \otimes_k k\{X\}$ is an isomorphism and therefore $(R \otimes_K R')^\delta \cong k\{X\}$. We claim that the induced isomorphism of σ -varieties $X \rightarrow \tilde{X}$ is an isomorphism of G -torsors: Let $Y \in \mathrm{GL}_n(R)$ with $\delta(Y) = AY$ and define Z and $Y' = Y \otimes Z \in \mathrm{GL}_n(R')$ as in the second step above. Then $(R \otimes_K R')^\delta = k\{\tilde{Z}, 1/\det(\tilde{Z})\}$ where $\tilde{Z} = Y^{-1} \otimes Y'$. As the isomorphism $(R \otimes_K R')^\delta \cong k\{X\}$

maps \tilde{Z} to Z , we see that \tilde{X} and X agree as σ -closed σ -subvarieties of GL_n . As G is acting on both by matrix multiplication from the left, we conclude that $X \rightarrow \tilde{X}$ is an isomorphism of G -torsors. \square

Let R' and R be σ -Picard-Vessiot rings for $\delta(y) = Ay$ and let B be a k - σ -algebra. We say that R and R' are isomorphic over B if there exists an isomorphism of $K \otimes_k B$ - $\delta\sigma$ -algebras between $R' \otimes_k B$ and $R \otimes_k B$. For the torsor X defined by Equation (4) we have $X(B) \neq \emptyset$ if and only if R and R' are isomorphic over B . So we obtain from Theorem 4.1:

COROLLARY 7.5. *Let R be a σ -Picard-Vessiot ring for $\delta(y) = Ay$ with σ -Galois group G . Then for a k - σ -algebra B , the set of isomorphism classes of σ -Picard-Vessiot rings for $\delta(y) = Ay$ that are isomorphic to R over B is in bijection with $H_\sigma^1(B/k, G)$.*

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